Math 216 — Second Midterm

12 November, 2012

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [16 points] Find real-valued solutions to each of the following, as indicated.

a. [8 points] Find the general solution to $y'' + 4y' + 5y = 3x - e^{-2x}$. The general solution to $y'' + 4y' + 5y = 0$ is $y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x)$.

Solution: There are two parts to the non-homogeneous term, $f_1(x) = 3x$ and $f_2(x) = -e^{-2x}$. Neither of these are part of the homogeneous solution to the problem, so we may use the Method of Undetermined Coefficients to guess for the first $y_{p1} = Ax + B$ and for the second $y_{p2} = Ce^{-2x}$. Plugging the first into the differential equation and setting it equal to f_1, we have

$$0 + 4A + 5Ax + 5B = 3x,$$

so that $A = 3/5$ and $B = -12/25$. Similarly, plugging in the second and setting it equal to f_2 we have

$$4C - 8C + 5C = -1,$$

so that $C = -1$. The general solution is thus $y = c_1e^{-2x}\cos(x) + c_2e^{-2x}\sin(x) + \frac{3}{5}\,x - \frac{12}{25} - e^{-2x}$.

b. [8 points] Find the solution to $y'' - y = 4e^x + 3\cos(x)$, $y(0) = 0$, $y'(0) = 1$. The general solution to $y'' - y = 0$ is $y = c_1e^x + c_2e^{-x}$.

Solution: Using the Method of Undetermined Coefficients, we again have two parts to the forcing and make two guesses for the parts of y_p. For the first part, we would guess $y_{p1} = Ae^x$, but this is part of the homogeneous solution, and so guess $y_{p1} = Axe^x$ instead. Then $y_{p1}'' = Axe^x + 2Ae^x$, and plugging in to get the first part of the right-hand side we have

$$Axe^x + 2Ae^x - Axe^x = 4e^x,$$

so that $A = 2$. For the second part we would guess $y_{p2} = B\cos(x) + C\sin(x)$, but because there are no odd derivatives in equation we know $C = 0$. Then, plugging in to get the cosine term on the right we have

$$-B\cos(x) - B\cos(x) = 3\cos(x),$$

so that $B = -3/2$. The general solution to the problem is

$$y = c_1e^x + c_2e^{-x} + 2xe^x - \frac{3}{2}\cos(x).$$

The initial condition $y(0) = 0$ requires that $c_1 + c_2 = 3/2$, and then

$$y'(x) = c_1e^x - c_2e^{-x} + 2e^x + 2xe^x + \frac{3}{2}\sin(x),$$

so that $y'(0) = c_1 - c_2 + 2 = 1$, or $c_1 - c_2 = -1$. Adding this to the previous equation, $c_1 = 1/4$, so that $c_2 = 5/4$. The final solution is thus $y = \frac{1}{4}\,e^x + \frac{5}{4}\,e^{-x} + 2xe^x - \frac{3}{2}\cos(x)$.

2. [14 points] Use the eigenvalue method to find a real-valued general solution to the matrix equation $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ if $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$.

Solution: We look for $\mathbf{x} = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ and λ are the eigenvectors and values of $\mathbf{A}$. Then we need

$$|\mathbf{A} - \lambda\mathbf{I}| = \left|\begin{pmatrix} 5-\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix}\right| = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17 = (\lambda-4)^2 + 1 = 0.$$

Thus $\lambda = 4 \pm i$. We can use either eigenvalue to find an eigenvector and then separate the real and imaginary parts of the resulting $\mathbf{x}$ to write the real-valued solution. With $\lambda = 4 + i$, $\mathbf{v}$ satisfies

$$\begin{pmatrix} 5-(4+i) & 2 \\ -1 & 3-(4+i) \end{pmatrix}\mathbf{v} = \mathbf{0},$$

so that, with $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$, we have two equations which are constant multiples of $(1-i)v_1 + 2v_2 = 0$. Letting $v_1 = 2$ we have $v_2 = -1 + i$, and a complex valued solution to the problem is

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} e^{4t}(\cos(t) + i\sin(t)) = \begin{pmatrix} 2\cos(t) \\ -\cos(t) - \sin(t) \end{pmatrix} e^{4t} + i\begin{pmatrix} 2\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^{4t}.$$

A real-valued general solution is therefore

$$\mathbf{x} = c_1\begin{pmatrix} 2\cos(t) \\ -\cos(t) - \sin(t) \end{pmatrix} e^{4t} + c_2\begin{pmatrix} 2\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^{4t}.$$

Note that an alternate formulation is to find $\mathbf{v} = \begin{pmatrix} -1 \mp i & 1 \end{pmatrix}$, so that

$$\mathbf{x} = c_1\begin{pmatrix} -\cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} e^{4t} + c_2\begin{pmatrix} -\cos(t) - \sin(t) \\ \sin(t) \end{pmatrix} e^{4t}.$$

3. [12 points] Consider the system

$$x_1' = a\,x_1 + 2x_2$$
$$x_2' = 4x_1 + x_2$$

(where a is a constant) and the linear constant-coefficient differential equation given in operator form by $(D^2 - 4D - 5)[y] = 0$.

a. [6 points] If the system and differential equation are equivalent, what is the value of a?

Solution: We know that if these are equivalent we will obtain the differential equation (with $y = x_1$ or $y = x_2$) if we eliminate x_2 or x_1 from the system. Writing the system in operator form, we have

$$\begin{aligned}(D-a)[x_1] - 2x_2 &= 0\\ -4x_1 + (D-1)[x_2] &= 0\end{aligned}$$

Operating on the first by $(D-1)$, multiplying the second by 2, and adding, we have

$$(D-1)(D-a)[x_1] - 8x_1 = (D^2 - (a+1)D + (a-8))[x_1] = 0.$$

Comparing coefficients with the operator for the given differential equation, we must have $a = 3$.

We can, of course, do this without using operator notation. From the first equation, $x_2 = \frac{1}{2}(x_1' - a\,x_1)$, so that $x_2' = \frac{1}{2}(x_1'' - a\,x_1')$. Plugging into the second equation, $\frac{1}{2}(x_1'' - a\,x_1') = 4x_1 + \frac{1}{2}(x_1' - a\,x_1)$, so $x_1'' - a\,x_1' = 8x_1 + x_1' - a\,x_1$, and

$$x_1'' - (a+1)x_1' + (a-8)x_1 = 0.$$

To no ones surprise this is the same as we found before.

b. [6 points] The general solution to $(D^2 - 4D - 5)[y] = 0$ is $y = c_1e^{5t} + c_2e^{-t}$. Using this, what is the general solution to the given system of differential equations?

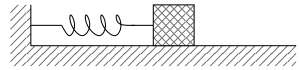
Solution: This is the solution for either x_1 or x_2. Taking it as x_1, we have $x_1 = c_1e^{5t} + c_2e^{-t}$. Then, from the first equation, with $a = 3$,

$$x_2 = \frac{1}{2}\left(x_1' - 3x_1\right) = \frac{1}{2}\left(5c_1e^{5t} - c_2e^{-t} - 3c_1e^{5t} - 3c_2e^{-t}\right) = c_1e^{5t} - 2c_2e^{-t}.$$

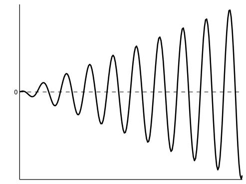
Alternately, if we take $x_2 = c_1e^{5t} + c_2e^{-t}$, we have from the second equation

$$x_1 = \frac{1}{4}\left(x_2' - x_2\right) = \frac{1}{4}\left(5c_1e^{5t} - c_2e^{-t} - c_1e^{5t} - c_2e^{-t}\right) = c_1e^{5t} - \frac{1}{2}\,c_2e^{-t}.$$

4. [12 points] Consider the differential equation $x'' + a\,x' + b\,x = A_0 \cos(\omega t)$, modeling dispacement x of the mass in the mass-spring system shown to the right. In this equation, a, b, A_0 and ω are constant parameters.

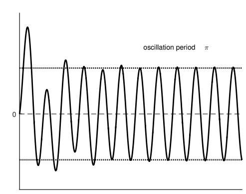


a. [6 points] If a representative graph of x as a function of time t is shown in the figure to the right, can you determine if any of a, b, A_0 or ω must be zero or must be non-zero? Must any of a, b, A_0 or ω be related in any way? Can you tell what value any of them must have?



Solution: This figure shows resonance, so we know $a = 0$, $b = \omega^2$ and $A_0 \neq 0$. We are unable to tell any specific values for these.

b. [6 points] If a representative graph of x as a function of time t is shown in the figure to the right, can you determine if any of a, b, A_0 or ω must be zero or must be non-zero? Must any of a, b, A_0 or ω be related in any way? Can you tell what value any of them must have?



Solution: This figure shows a transient motion followed by a steady-state oscillation, so we know $a, b > 0$ and $A_0 \neq 0$. The frequency ω determines the period of the steady state oscillation, so $\omega = 2$.

5. [16 points] Identify each of the following as true or false. Give a one-sentence explanation for your response in each case.

a. [4 points] Euler's method applied to the system $\mathbf{x}' = \begin{pmatrix} t & 0 \\ 1 & t^2 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives, after 2 steps with $h = 0.5$, $\mathbf{x}(1) \approx \begin{pmatrix} 0 \\ 1.25 \end{pmatrix}$.

True $\boxed{\text{False}}$

Solution: Because $\mathbf{x}'(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{0}$, Euler's method gives $\mathbf{x}(0.5) \approx \mathbf{x}(0)$; then $\mathbf{x}'(0.5) \approx \begin{pmatrix} 0.5 & 0 \\ 1 & 0.25 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.25 \end{pmatrix}$, and $\mathbf{x}(1) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0.5\begin{pmatrix} 0 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0 \\ 1.125 \end{pmatrix}$.

b. [4 points] Given that $\mathbf{x}_1 = \begin{pmatrix} e^t \\ 3e^t \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 2e^t \\ 6e^t \end{pmatrix}$ are solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for some 2×2 matrix $\mathbf{A}$, a general solution is $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

True $\boxed{\text{False}}$

Solution: The two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are not linearly independent, so this cannot be a general solution.

c. [4 points] If $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, $\mathbf{x}_n(t)$ are solutions to a system of n linear first-order differential equations, and if $\mathbf{x}_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\mathbf{x}_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $\mathbf{x}_n(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, then a general solution to the system is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$.

$\boxed{\text{True}}$ False

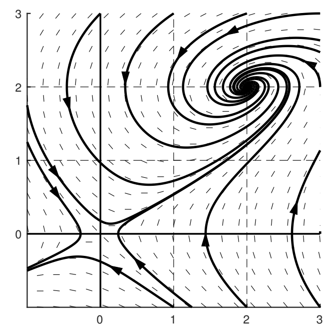
Solution: Note that $W[\mathbf{x}_1(0), \ldots, \mathbf{x}_n(0)] = 1$; thus the $\mathbf{x}_j$ are linearly independent and the general solution is as given.

d. [4 points] If one or more of the eigenvalues of the constant matrix $\mathbf{A}$ are zero, the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has no solution.

True $\boxed{\text{False}}$

Solution: Because of the existence theorem we know that there is a solution.

6. [15 points] Consider the phase portrait shown to the right, which is for a system $x' = f(x, y)$, $y' = g(x, y)$.



a. [3 points] Explain why this system must be nonlinear.

Solution: If f and g are linear there is only one critical point; this system has two (at $(0, 0)$ and $(2, 2)$), and must therefore have some nonlinearity.
Alternately, we know for a linear system we can rewrite this as $\mathbf{x}' = \mathbf{A}\mathbf{x}$ and solve to find a solution that will have nodal, saddle point, or spiral point behavior centered at a critical point. Here we see two types of behavior: a spiral at $(2, 2)$ and a saddle at $(0, 0)$, which isn't possible for a linear system.

b. [8 points] Now suppose that the system is given by

$$x' = a\,y - y^2$$
$$y' = x + b\,y$$

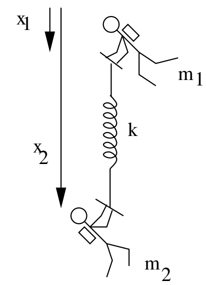
What are the parameters a and b? Why?

Solution: The critical points shown in the phase portrait are $(0, 0)$ and $(2, 2)$. We can find a and b by using the latter of these. At critical points $x' = y' = 0$; thus, if $y = 2$ the first equation gives $0 = 2a - 4$, so that $a = 2$. Similarly in the second, with $x = 2$ and $y = 2$, $0 = 2 + 2b$, so that $b = -1$.

c. [4 points] For each of the critical points of the system shown in the figure above, indicate whether it appears to be unstable, stable, or asymptotically stable; and whether it is a node, saddle point, center or spiral point.

Solution: The point $(0, 0)$ is a(n unstable) saddle point; the point $(2, 2)$ is an asymptotically stable spiral point (a spiral sink).

7. [15 points] The figure to the right shows two (hypothetical) skydivers, with a spring connecting them. We assume that the mass of the first, m_1, is less than the mass of the second, m_2. The distances that each has fallen are x_1 and x_2, and the spring constant is k. Let L be the equilibrium length of the spring. Then the system is modeled as



$$x_1'' = \frac{k}{m_1}(-x_1 + x_2) + \left(g - \frac{kL}{m_1}\right)$$
$$x_2'' = \frac{k}{m_2}(x_1 - x_2) + \left(g + \frac{kL}{m_2}\right).$$

a. [3 points] If we write this as a matrix equation $\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{f}$, what are $\mathbf{x}$, $\mathbf{A}$ and $\mathbf{f}$?

Solution: These are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -k/m_1 & k/m_1 \\ k/m_2 & -k/m_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} g - kL/m_1 \\ g + kL/m_2 \end{pmatrix}.$$

b. [4 points] Now suppose that we're interested in finding the solution to the homogeneous problem associated with this system. If we take $\mathbf{x} = \mathbf{v}e^{\omega t}$, what equation must $\mathbf{v}$ and ω satisfy? How are $\mathbf{v}$ and ω related to the matrix $\mathbf{A}$ that you found above?

Solution: Plugging $\mathbf{x} = \mathbf{v}e^{\omega t}$ into the homogeneous problem $\mathbf{x}'' = \mathbf{A}\mathbf{x}$, we get $\omega^2\mathbf{v}e^{\omega t} = \mathbf{A}\mathbf{v}e^{\omega t}$. We may divide out the exponential to obtain $\omega^2\mathbf{v} = \mathbf{A}\mathbf{v}$, so that $\omega^2 = \lambda$, the eigenvalues of $\mathbf{A}$, and $\mathbf{v}$ are the corresponding eigenvectors.

c. [8 points] Now suppose that the eigenvalues and eigenvectors of the matrix $\mathbf{A}$ you found in (a) are $\lambda_1 = 0$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -4$ with $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Write the complementary homogeneous solution to your system.

Solution: From the work above, if $\lambda = 0$ we know that $\omega = 0$ (twice), and we get the two solutions $\mathbf{x}_1 = \mathbf{v}_1$ and $\mathbf{x}_2 = \mathbf{v}_1 t$. If $\lambda = -4$ we have $\omega = \pm 2i$, so that we gain the two additional solutions $\mathbf{x}_3 = \mathbf{v}_2\cos(2t)$ and $\mathbf{x}_4 = \mathbf{v}_2\sin(2t)$. The general solution is therefore

$$\mathbf{x} = (c_1 + c_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (c_3\cos(2t) + c_4\sin(2t))\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$