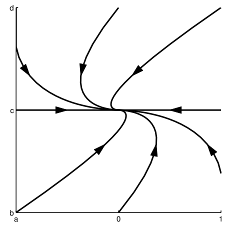
Math 216 — Final Exam

14 December, 2012

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] For each of the following the given figure is a phase portrait for a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant 2×2 matrix. For each select the correct characterization of the eigenvalues of $\mathbf{A}$ and fill in the requested information about an eigenvector of this matrix.

a. [4 points]



The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2;$ $\quad$ $\lambda_1 = -1,\ \lambda_2 = 2;$

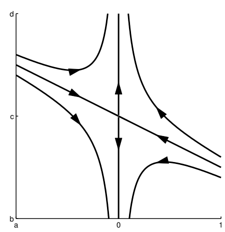
$\boxed{\lambda_1 = -1,\ \lambda_2 = -2;}$ $\quad$ $\lambda_{1,2} = 1 \pm i;$

$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): $\mathbf{v} = \begin{pmatrix}1 & 0\end{pmatrix}^T$

Solution: There is one obvious straight line trajectory (the x-axis), so the eigenvalues and vectors must be real, and one eigenvector must be $\mathbf{v} = \begin{pmatrix}1 & 0\end{pmatrix}^T$. Both eigenvalues are negative because all trajectories approach the origin.

b. [4 points]



The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2;$ $\quad$ $\boxed{\lambda_1 = -1,\ \lambda_2 = 2;}$

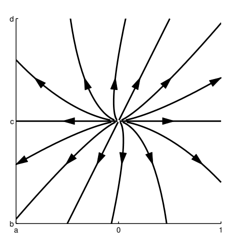
$\lambda_1 = -1,\ \lambda_2 = -2;$ $\quad$ $\lambda_{1,2} = 1 \pm i;$

$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): $\mathbf{v} = \begin{pmatrix}0 & 1\end{pmatrix}^T$

Solution: This is a saddle point, so there are two real eigenvalues, one positive and one negative. The positive eigenvalue corresponds to the vertical trajectories along the eigenvector $\mathbf{v} = \begin{pmatrix}0 & 1\end{pmatrix}^T$.

c. [4 points]



The eigenvalues of $\mathbf{A}$ could be (circle one):

$\boxed{\lambda_1 = 1,\ \lambda_2 = 2;}$ $\quad$ $\lambda_1 = -1,\ \lambda_2 = 2;$

$\lambda_1 = -1,\ \lambda_2 = -2;$ $\quad$ $\lambda_{1,2} = 1 \pm i;$

$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): $\mathbf{v} = \begin{pmatrix}1 & 0\end{pmatrix}^T$

Solution: This is a nodal source, so there are two real eigenvalues, both positive. One of eigenvectors lies along the x-axis, so it is $\mathbf{v} = \begin{pmatrix}1 & 0\end{pmatrix}^T$, and is associated with the smaller of the eigenvalues.

2. [14 points] A model for a population that is susceptible to a disease is the SI (Susceptible, Infected) model. With a few simplifying assumptions, we may model smallpox infections in a population with the SI model

$$\begin{aligned} S' &= -4SI + k(1 - S - I) \\ I' &= 4SI - I, \end{aligned}$$

where S is the fraction of the total population that is susceptible to smallpox and I is the fraction who are infected by the disease. (The remainder of the population is recovered.) We shall consider this with $k = 2$, in which case the equilibrium solutions to the system are $(S, I) = (1, 0)$ and $(S, I) = (1/4, 1/2)$.

a. [5 points] Find the linearization of this system at the critical point $(1, 0)$. Solve the linear system that you obtain.

Solution: The Jacobian for the system is $J = \begin{pmatrix} -4I - 2 & -4S - 2 \\ 4I & 4S - 1 \end{pmatrix}$, which at $(1, 0)$ is $J(1, 0) = \begin{pmatrix} -2 & -6 \\ 0 & 3 \end{pmatrix}$. Thus, if $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a small displacement from $(1, 0)$, we have $\mathbf{u}' = \begin{pmatrix} -2 & -6 \\ 0 & 3 \end{pmatrix} \mathbf{u}$. Eigenvalues of $J(1, 0)$ are $\lambda = -2, 3$; for $\lambda = -2$, $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ and for $\lambda = 3$, $\mathbf{v} = \begin{pmatrix} -6 & 5 \end{pmatrix}^T$. Thus

$$\mathbf{u} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -6 \\ 5 \end{pmatrix} e^{3t}.$$

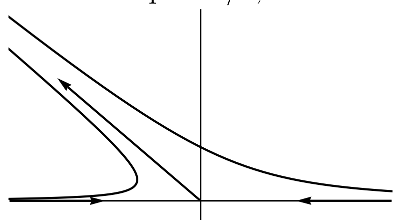
b. [4 points] Find the linearization of this system at the critical point $(1/4, 1/2)$. Determine the type of critical point this is (that is, whether it is a node, saddle or spiral point, and its stability).

Solution: Using the Jacobian from above and again taking $\mathbf{u}$ to be the equilibrium solution we have $\mathbf{u}' = \begin{pmatrix} -4 & -3 \\ 2 & 0 \end{pmatrix} \mathbf{u}$. Eigenvalues of the coefficient matrix are given by $(-4 - \lambda)(-\lambda) + 6 = \lambda^2 + 4\lambda + 6 = (\lambda + 2)^2 + 2 = 0$, so that $\lambda = -2 \pm i\sqrt{2}$. Thus this is an asymptotically stable spiral point.

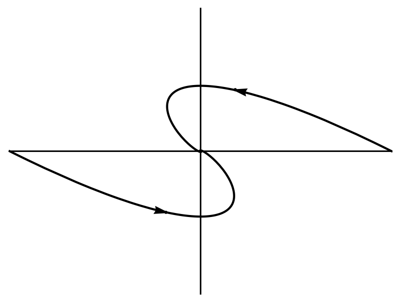
Problem 2, continued

c. [5 points] Based on your work in (a) and (b), sketch a qualitatively reasonable phase portrait (including equilibrium solutions and representative trajectories) for this system on the domain $0 \le S \le 1$ and $0 \le I \le 1$. Based on your work on this part of the problem and on (a) and (b), how do you expect the populations of susceptible and infected individuals to evolve if we start with the initial condition $(S, I) = (0.8, 0.01)$?

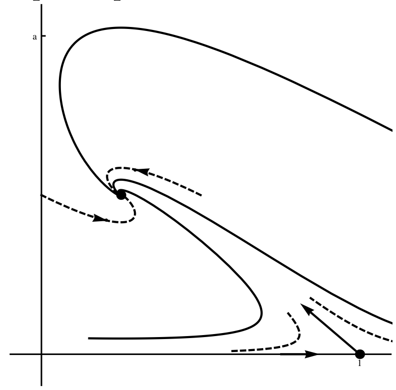
Solution: At $(1, 0)$ we have a saddle point with trajectories converging along the x (S) axis, and diverging along a line with slope $-5/6$, as shown in the figure below.



At $(1/4, 1/2)$ the direction of the spiral is suggested by the Jacobian: if we start directly above the equilibrium point, $\mathbf{u}'$ is proportional to $J(1/4, 1/2)\begin{pmatrix}0 & 1\end{pmatrix}^T = \begin{pmatrix}-4 & 0\end{pmatrix}^T$, so trajectories are moving counter-clockwise around, and converging to, the equilibrium point. This gives the behavior shown below.



Combining these, we have the phase plane shown below.



Thus we expect a trajectory starting at $(0.8, 0.01)$ to initially move to the right and up (that is, S and I both increase), then to have decreasing S and increasing I until they are on the order of $S = 1/4$ with $I > 1/2$. The populations will then oscillate around the values $S = 1/4$ and $I = 1/2$, converging to the equilibrium point. In the long term we expect to see $(S, I) = (1/4, 1/2)$.

3. [12 points] Use Laplace transforms to solve the initial value problem

$$y'' + y = \begin{cases} 1 & t < 2 \\ 0 & t \geq 2 \end{cases}, \quad y(0) = 3, \quad y'(0) = 0.$$

Solution: Note that we can write the differential equation as $y'' + y = 1 - u(t-2)$. Transforming both sides of this, we have

$$\begin{aligned} \mathfrak{L}\{y'' + y\} &= \mathfrak{L}\{1 - u(t-2)\}, \quad \text{or} \\ (s^2 Y - 3s) + Y &= \frac{1}{s} - \frac{e^{-2s}}{s}, \end{aligned}$$

where $Y = \mathfrak{L}\{y\}$ is the Laplace transform of $y(t)$. Thus

$$Y = \frac{3s}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)}.$$

To invert this we need the inverse transforms of each term, but note that the third term will follow directly from the second from the inverse transform rule for $e^{-as}F(s)$. The first term inverts using the transform for $\cos kt$, so that

$$\mathfrak{L}^{-1}\{\frac{3s}{s^2+1}\} = 3\cos t.$$

Then we note that $\mathfrak{L}^{-1}\{\frac{1}{s^2+1}\} = \sin t$, so that by the transform rule for $F(s)/s$ we have

$$\mathfrak{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \int_0^t \sin\tau\, d\tau = 1 - \cos t.$$

Finally, using this we have

$$\mathfrak{L}^{-1}\{\frac{e^{-2s}}{s(s^2+1)}\} = u(t-2)(1 - \cos(t-2)).$$

Thus the solution is

$$\begin{aligned} y(t) = \mathfrak{L}^{-1}\{Y\} &= 3\cos t + (1 - \cos t) - u(t-2)(1 - \cos(t-2)) \\ &= 1 + 2\cos t - u(t-2)(1 - \cos(t-2)). \end{aligned}$$

4. [12 points] Fill in the blanks for each of the following inverse Laplace transforms. Show enough work to indicate how you obtained your answer.

a. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{1}{s^2+8s+16}\right\} = \mathfrak{L}^{-1}\left\{\dfrac{1}{(s+4)^2}\right\} = \underline{\quad t\,e^{-4t} \quad}$

Solution: Note that $\mathfrak{L}^{-1}\{1/s^2\} = t$. Thus, by the transform rule $\mathfrak{L}\{e^{at}f(t)\} = F(s-a)$, we have $\mathfrak{L}^{-1}\{\frac{1}{(s+4)^2}\} = t\,e^{-4t}$.

b. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{3s+1}{(s+2)^2+16}\right\} = 3\,e^{-2t}\cos(4t) + \underline{\quad -5\,e^{-2t}\sin(4t)/4 \quad}$

Solution: We note that $\frac{3s+1}{(s+2)^2+16} = \frac{3(s+2)-5}{(s+2)^2+16}$. The first term in the numerator gives the first term in the expression provided, and we see that the second then gives $\mathfrak{L}^{-1}\{-\frac{5}{(s+2)^2+16}\} = -\frac{5}{4}\,e^{-2t}\sin(4t)$.

c. [4 points] $\mathfrak{L}^{-1}\left\{\dfrac{2-e^{-4s}}{s^2+9}\right\} = \dfrac{2}{3}\sin(3t) + \underline{-u(t{-}4)\,\sin(3(t{-}4))/3}$

Solution: The first term in the inverse transform tells us $f(t) = \frac{1}{3}\sin(3t)$ for $F(s) = \frac{1}{s^2+9}$. The answer then is derived easily from the rule for $e^{-as}F(s)$: $\mathfrak{L}^{-1}\{-\frac{e^{-4s}}{s^2+9}\} = -\frac{1}{3}\,u(t-4)\,\sin(3(t-4))$.

5. [14 points] Solve each of the following, as indicated.

a. [7 points] Find the general solution to $xz'(x) = z(x) + 5x^4$.

Solution: This is a first-order linear equation, so we may use the method of integrating factors. We rewrite the equation as $z' - \frac{1}{x}\,z = 5x^3$ to see that the integrating factor is $\mu(x) = e^{-\int 1/x\,dx} = 1/x$, so that we have $(z/x)' = 5x^2$. Integrating both sides, $z/x = \frac{5}{3}\,x^3 + C$, so that $z = \frac{5}{3}\,x^4 + Cx$.

b. [7 points] Find the solution to the initial value problem $(1 + x)y' = 3y^2$, $y(0) = 2$.

Solution: This is a nonlinear but separable first-order equation. Separating variables, we have $\int y'/y^2\,dy = \int 3/(1 + x)\,dx$, so that

$$-\frac{1}{y} = 3\ln|1 + x| + C,$$

and

$$y = -\frac{1}{3\ln|1 + x| + C}.$$

The initial condition $y(0) = 2$ requires that $C = -1/2$, so that

$$y = -\frac{1}{3\ln|1 + x| - \frac{1}{2}} = \frac{2}{1 - 6\ln|1 + x|}.$$

6. [14 points] Consider the differential equation $\frac{d^2z}{dt} + 2\frac{dz}{dt} + z = 0$.

a. [6 points] Find a general real-valued solution to this differential equation.

Solution: Let $z = e^{rt}$. Then $r^2 + 2r + 1 = (r + 1)^2 = 0$, so $r = -1$ twice, and we know the general solution is

$$z = c_1 e^{-t} + c_2 t e^{-t}.$$

b. [3 points] Rewrite the equation as a linear system of first order equations, in the form $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$.

Solution: Let $x_1 = z$ and $x_2 = z'$. Then we have $x_1' = x_2$ and $x_2' = -x_1 - 2x_2$, or

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

c. [5 points] Use your solution from (a) to write the general solution for $\mathbf{x}$ in your system in (b).

Solution:

$$\mathbf{x} = \begin{pmatrix} z \\ z' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} t \\ 1 - t \end{pmatrix} e^{-t}.$$

(As a side note, we observe that this is of the form $c_1\mathbf{v}_1 e^{-t} + c_2(\mathbf{v}_2 + \mathbf{v}_1\,t)e^{-t}$—that is, that the repeated root results in a term linear in t—much as we might expect from our result in (a).)

7. [10 points] Values from a cubic polynomial $f(y)$ are shown in the table below.

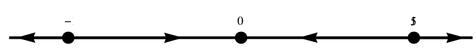
$y =$	-2	-1	0	1	2
$f(y) =$	-3.5	1.25	0	-1.25	3.5

a. [5 points] How many equilibrium solutions are there for the differential equation $\frac{dy}{dx} = f(y)$, where $f(y)$ is represented in the table above? What might these solutions be?

Solution: There is an equilibrium at $y = 0$; because $f(y)$ is continuous we know that there must be two other equilibria, one between $y = -2$ and $y = -1$ and the other between $y = 1$ and $y = 2$. We therefore know that there are three equilibrium solutions, and might guess their values to be $y = 0$ and $y = \pm 1.5$.

b. [5 points] Identify the stability of each of the equilibrium solutions you found in (a), and sketch a phase diagram for the differential equation.

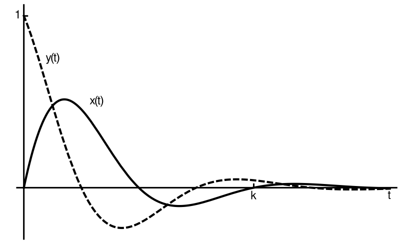
Solution: We can see that $y = \pm 1.5$ are unstable, while $y = 0$ is stable. The phase diagram is shown below.



8. [12 points] The figure to the right shows the solution to the system of first-order equations

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & a \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where a is a constant. In the figure, x is given by the solid curve and y by the dashed curve, and k is a constant we specify later.



a. [4 points] Given these solution curves, is a positive, negative, zero, or can we not tell?

Solution: Because the solution curves are oscillatory, we know that the eigenvalues of the coefficient matrix must be complex. The eigenvalues satisfy the equation

$$\left| \begin{pmatrix} -1-\lambda & a \\ -2 & -1-\lambda \end{pmatrix} \right| = (\lambda+1)^2 + 2a = 0.$$

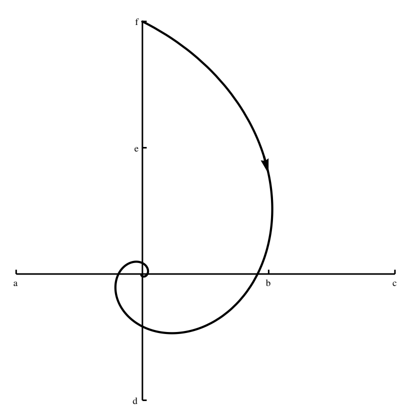
Thus $\lambda = -1 \pm \sqrt{-2a}$. This will have complex solutions for λ if $a > 0$. Thus we know that $a > 0$.

b. [4 points] If we know that $k = \pi$, what is a?

Solution: This tells us that the pseudoperiod of the decaying oscillation seen is π, so that $\lambda = \alpha \pm 2i$. From (a), we have $\lambda = -1 \pm i\sqrt{2a}$, so $a = 2$.

c. [4 points] Sketch a phase portrait in the x-y plane for this system, showing the solution trajectory illustrated in the figure given in the problem statement.

Solution: The trajectory shown starts at $(x, y) = (0, 1)$. We know that this is spiral trajectory that will approach the origin, and initially x increases and y decreases, so it must be a clockwise spiral. (This is also evident if we take $(x, y) = (0, 1)$ and calculate $(x, y)' = (2, -1)$ from the system.) This gives the phase portrait shown below.



Do not use this page for work on the problems on the final exam. Any work you do here will not be considered when grading your exam.

Some formulas which may or may not prove useful

- Euler's, improved Euler and Runge Kutta methods iteration steps for $x' = f(t,x)$ (x and f scalar or vector):
  Euler: $x_{j+1} = x_j + hf(t_j, x_j)$
  Improved Euler: $x_{j+1} = x_j + \frac{h}{2}(f(t_j, x_j) + f(t_{j+1}, u))$, where $u = x_j + hf(t_j, x_j)$
  Runge Kutta: $x_{j+1} = x_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, where $k_1 = f(t_j, x_j)$, $k_2 = f(t_j + \frac{h}{2}, x_j + \frac{h}{2}k_1)$, $k_3 = f(t_j + \frac{h}{2}, x_j + \frac{h}{2}k_2)$, and $k_4 = f(t_{j+1}, x_j + hk_3)$
- Some integration formulas:
  $\int 1/\sqrt{a^2 - x^2}\, dx = \arcsin(x/a) + C$ $\quad\int 1/(a^2 + x^2)\, dx = \arctan(x/a)/a + C$
  $\int \sin^2(x)\, dx = x/2 - \sin(2x)/4 + C$ $\quad\int \cos^2(x)\, dx = x/2 + \sin(2x)/4 + C$
  $\int \tan^2(x)\, dx = \tan(x) - x + C$.
- Laplace transforms:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

$f(t)$	$F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$f(t) * g(t)$	$F(s) \cdot G(s)$
$-tf(t)$	$F'(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$
$u(t-a)f(t-a)$	$e^{-as}F(s)$