Math 216 — First Midterm

13 October, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find the general solution to $y' = \sin(t) - \dfrac{\sin(t)}{\cos(t)}\, y$.

b. [7 points] Find a solution, explicit or implicit, for y, if

$$y' = \frac{1+\sin(t)}{1+\cos(y)}, \qquad y(\pi) = 0.$$

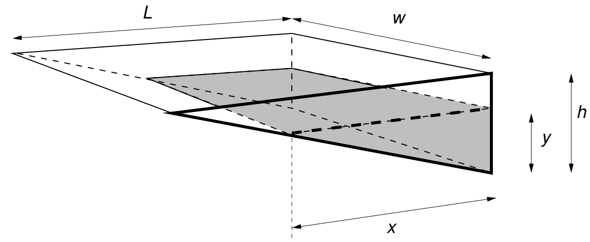
2. [16 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [8 points] The general solution to the system $x_1' = 2x_1 + 3x_2$, $x_2' = x_1 + 4x_2$

b. [8 points] The solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

3. [12 points] Consider an abandoned zero-entry pool as suggested by the figure to the right, below. (The front face of the pool is shown with bold lines.) In the figure, w, L and h are the fixed dimensions of the pool, and x and y characterize the part that is filled with water. The corresponding volume of the filled section is $V = \frac{1}{2}\, w\, x\, y$.

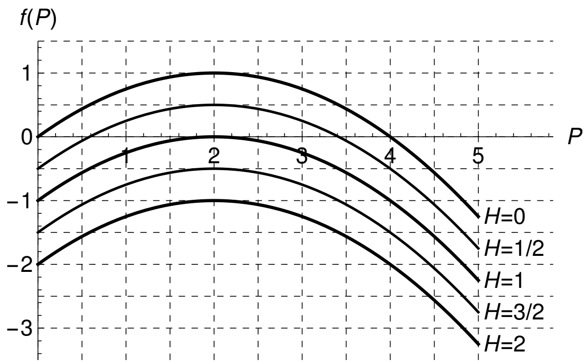
a. [8 points] If the pool slowly evaporates at a volumetric rate proportional to its top surface area, write a differential equation for the volume of the water in the pool.



b. [4 points] Solve your equation from (**a**) with the initial condition $V(0) = V_0$. At what time is the pool finally empty? *(If you are unable to find an equation in (a), you may proceed with the equation $V' = -k\sqrt{V}$.)*

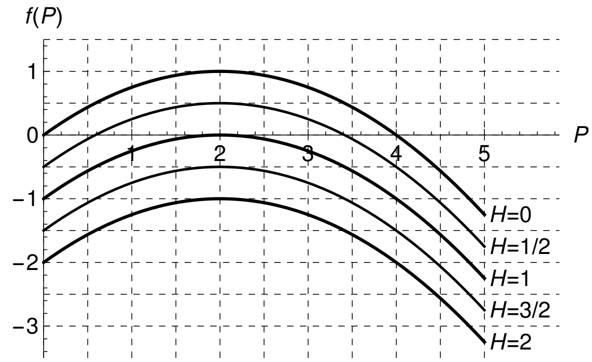
4. [18 points] A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.

a. [6 points] Plot phase lines for this equation when $H = 0$, $H = 1$ and $H = 2$. For each, identify all equilibrium solutions and their stability.

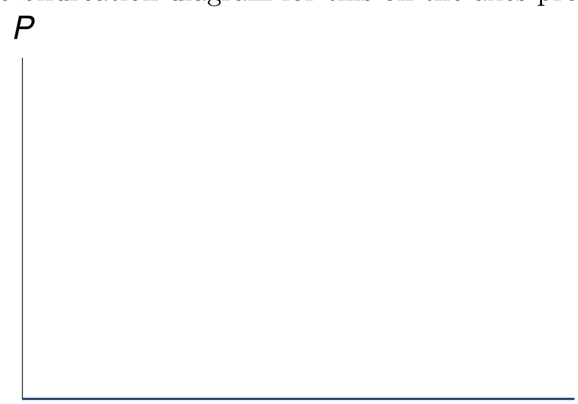


b. [5 points] Sketch qualitatively accurate solution curves for the case $H = 0$. Include enough initial conditions to show all solution behaviors.

Problem 4, continued. Instructions are reproduced here:
A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.



c. [4 points] This problem and your work on it provide an example of a model with a bifurcation. Draw the bifurcation diagram for this on the axes provided below.



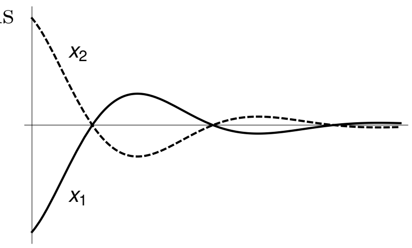
d. [3 points] Explain what your work in the preceding indicates about the long-term survival of the harvested population (fish).

5. [16 points] Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

for some real-valued, constant, 2×2 matrix $\mathbf{A}$. Suppose that one solution to (1) is $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$. Identify each of the following as true or false, by circling "True" or "False" as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer.

a. [4 points] A possible component plot of solutions to (1) is



True False

b. [4 points] The general solution to (1) could be $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$.

True False

c. [4 points] The equation $\mathbf{A}\mathbf{w} = -\mathbf{w}$ has infinitely many solutions $\mathbf{w}$.

True False

d. [4 points] An eigenvalue of the matrix $\mathbf{A}$ could be $\lambda = 1 + i$.

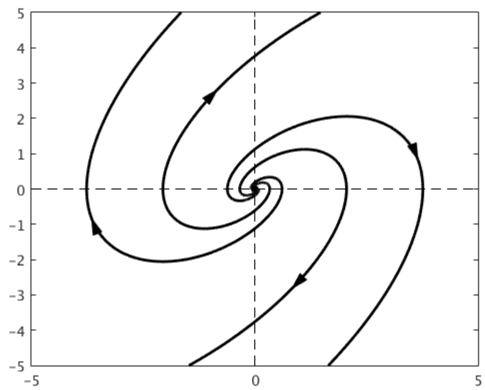
True False

6. [12 points] Consider the system of equations

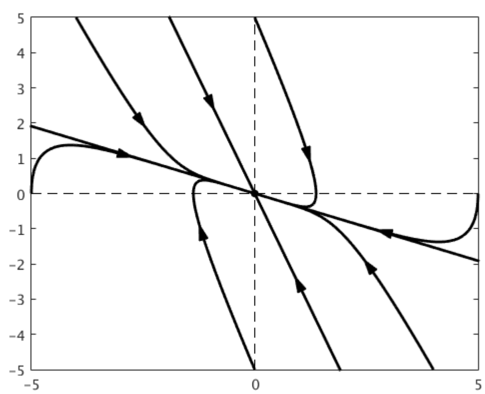
$$x_1' = x_2$$
$$x_2' = -x_1 + \alpha x_2,$$

where α is a real-valued constant. For each of the phase portraits shown below, indicate all values for α that could result in this sytem having a phase portrait of that type and with the indicated stability. If it is not possible, write "**not possible**" and give a short explanation why.

a. [6 points]



b. [6 points]



7. [12 points] The equation of motion for a damped, nonlinear pendulum is

$$\theta'' + c\,\theta' + k\,\sin(\theta) = 0,$$

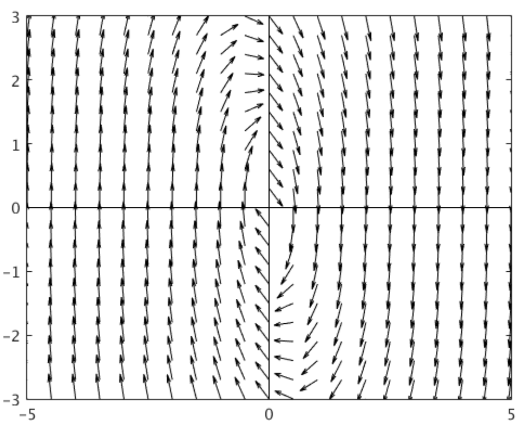
where θ is the angle the pendulum makes with its midline, and c and k are positive (non-zero) real-valued constants.

a. [4 points] Rewrite this as a system of equations in $x = \theta$ and $y = \theta'$.

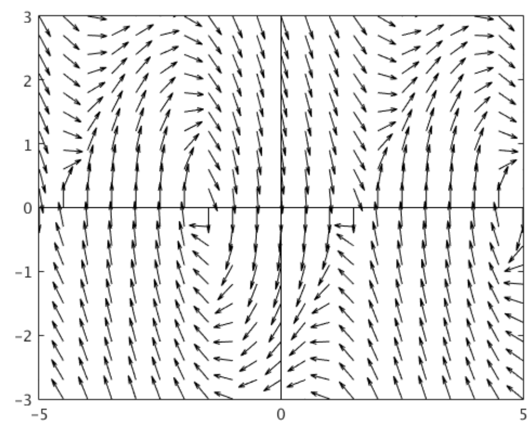
b. [4 points] Find all critical points for your system from (a).

c. [4 points] Which of the direction fields below could be that of your system in (a)? Why?

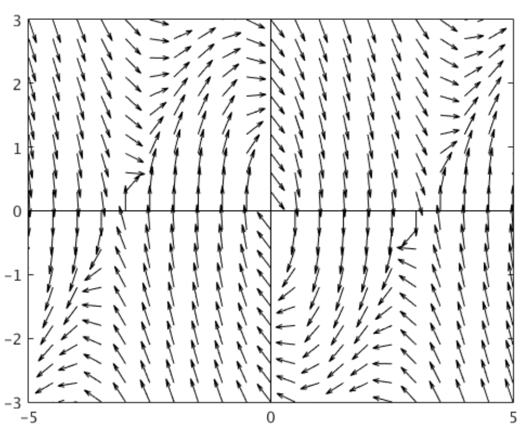
i.



ii.



iii.



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