Math 216 — Second Midterm

17 November, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$
C.	$e^{ct}f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$

1. [16 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [8 points] Find the general solution to $y'' + 4y' + 13y = 26t$.

b. [8 points] Solve $y'' + 6y' + 8y = e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

2. [14 points] Find each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $\mathcal{L}\{f(t)\}$, if $f(t) = \begin{cases} 1-t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$

b. [7 points] $Y(s) = \mathcal{L}\{y(t)\}$, if $y'' + 9y = u_\pi(t)\cos(4(t-\pi))$, $y(0) = 1$, $y'(0) = 2$.

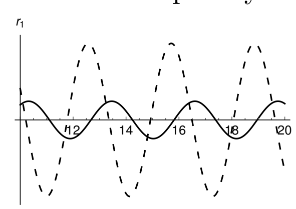
3. [15 points] A chemical reaction with two reagents (chemicals) in amounts r_1 and r_2 that may be converted from one to the other may be modeled the system of first-order differential equations

$$r_1' = -3r_1 + 9r_2$$
$$r_2' = k\,r_1 - r_2 + f(t),$$

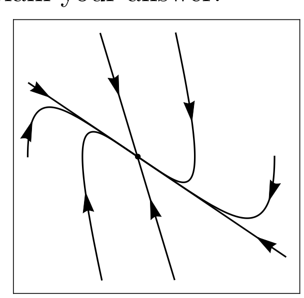
where $f(t)$ is the rate at which the second reagent is being added to the reaction and k is a constant.

a. [5 points] Write down the second-order linear equation which has r_1 as its solution.

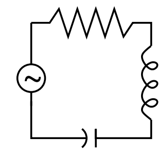
b. [5 points] If $f(t) = \cos(\omega t)$ is the dashed curve in the figure below, for what values of k, if any, could the long-term behavior of r_1 be that shown by the solid curve? Explain your answer.



c. [5 points] If $f(t) = A_0$, a constant, for what values of k, if any, could the phase portrait for this system be similar to that shown in the figure below? Explain your answer.

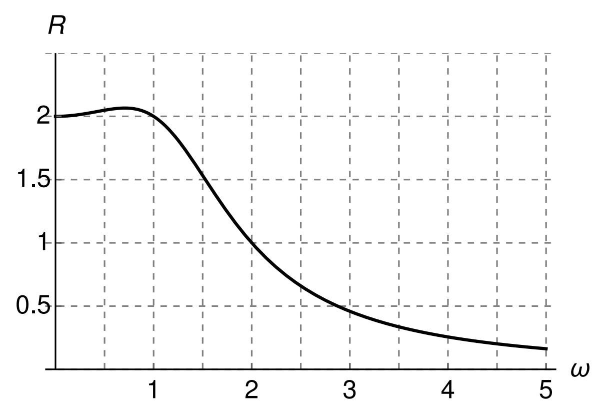


4. [13 points] Consider the RLC circuit shown to the right, below. This is modeled by $y'' + k\,y' + 2y = g(t)$, where $g(t)$ is the derivative of the input voltage and $0 < k < 2\sqrt{2}$ is proportional to the resistance of the resistor.



a. [9 points] If $g(t) = 4\cos(t)$, find the steady state response to the input. Write your answer in the form $R\cos(t - \alpha)$.

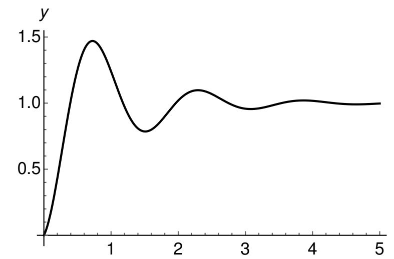
b. [4 points] The amplitude of the steady state response to the forcing $g(t) = 4\cos(\omega t)$ is shown below, as a function of ω. What is the value of k in the equation? Why?



5. [14 points] For the first two of the following, identify each as true or false, by circling "True" or "False" as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer. For the last give a short answer explaining the indicated question.

a. [4 points] For some constant ω and k, a solution to the mechanical system $y'' + 2y' + ky = \cos(\omega t)$ could be that shown to the right.

True False



b. [4 points] Let $F(s) = \frac{s^2+1}{s^2+3s+5}$. There is some piecewise continuous function $f(t)$, of exponential order, for which $\mathcal{L}\{f(t)\} = F(s)$.

True False

c. [6 points] Your friends Anna and Andrew are solving the two problems $y'' + 0.1y' + y = 0$, $y(0) = 0$, $y'(0) = 1$ and $z'' + 0.1z' + z = \delta(t-3)$, $z(0) = 0$, $z'(0) = 0$. Anna thinks that $z(t) = y(t-3)$, while Andrew thinks they are different. Explain why they are both partly correct.

6. [14 points] Find solutions to each of the following, as indicated.

a. [7 points] Find the general solution to $y'' + y' = \dfrac{1}{1 + e^t}$. *(Hint:* $\int \frac{1}{1+e^t}\, dt = t - \ln(1 + e^t)$.*)*

b. [7 points] Find the solution to $y'' - y' = u_2(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$

7. [14 points] Recall the linearized version of our laser model,

$$u' = -\gamma(Au + v)$$
$$v' = (A - 1)\,u.$$

Consider the case of constant forcing (A = a constant) and the initial conditions $u(0) = u_0$, $v(0) = 0$.

a. [6 points] Find a system of algebraic equations for $U(s) = \mathcal{L}\{u(t)\}$ and $V(s) = \mathcal{L}\{v(t)\}$.

b. [4 points] Solve your system from **(a)** to find $U(s)$ and $V(s)$.
(If you are unable to solve (a), consider the system $(s+a)U+bV = u_0$, $-cU+(s+a)V = 0$.*)*

c. [4 points] Recall that in the lab we solved the characteristic equation $\lambda^2+\gamma A\lambda+\gamma(A-1) = 0$, finding $\lambda^2 + \gamma A\lambda + \gamma(A-1) = (\lambda + \frac{1}{2}\,\lambda A)^2 + (-\frac{1}{4}\gamma^2A^2 + \gamma(A-1)) = (\lambda + \mu)^2 + \nu^2 = 0$ (so that $\lambda = -\mu \pm i\nu$). Use this to rewrite your solutions in **(b)** in terms of μ and ν. Find $u(t) = \mathcal{L}^{-1}\{U(s)\}$ and $v(t) = \mathcal{L}^{-1}\{V(s)\}$ in terms of μ and ν.
(If you are stuck, assume the denominator of your U and V is of the form $(s + a)^2 + b^2$.*)*

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