This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty.** In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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### Some Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $1$</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>2. $e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>3. $t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>4. $\sin(at)$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>5. $\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>6. $u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
</tr>
<tr>
<td>7. $\delta(t-c)$</td>
<td>$e^{-cs}$</td>
</tr>
</tbody>
</table>

| A. $f'(t)$             | $sF(s) - f(0)$             |
| A.1 $f''(t)$           | $s^2F(s) - sf(0) - f'(0)$  |
| A.2 $f^{(n)}(t)$       | $s^nF(s) - \cdots - f^{(n-1)}(0)$ |

| B. $t^n f(t)$          | $(-1)^n F^{(n)}(s)$       |
| C. $e^{ct} f(t)$       | $F(s-c)$                  |
| D. $u_c(t) f(t-c)$     | $e^{-cs} F(s)$            |
| E. $f(t)$ (periodic with period $T$) | $\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) \, dt$ |
1. [16 points] Find real-valued solutions for each of the following, as indicated. (Note that minimal partial credit will be given on this problem.)
   
a. [8 points] Find the general solution to $y'' + 4y' + 13y = 26t$.

b. [8 points] Solve $y'' + 6y' + 8y = e^{-2t}$, $y(0) = 0$, $y'(0) = 0$. 
2. [14 points] Find each of the following. (Note that minimal partial credit will be given on this problem.)

a. [7 points] \( \mathcal{L}\{f(t)\} \), if 
\[
\begin{cases}
1 - t, & 0 \leq t < 1 \\
0, & \text{otherwise}
\end{cases}
\]

b. [7 points] \( Y(s) = \mathcal{L}\{y(t)\} \), if 
\[
y'' + 9y = u_\pi(t) \cos(4(t - \pi)), \quad y(0) = 1, \ y'(0) = 2.
\]
3. [15 points] A chemical reaction with two reagents (chemicals) in amounts \( r_1 \) and \( r_2 \) that may be converted from one to the other may be modeled the system of first-order differential equations
\[
\begin{align*}
r'_1 &= -3r_1 + 9r_2 \\
r'_2 &= kr_1 - r_2 + f(t),
\end{align*}
\]
where \( f(t) \) is the rate at which the second reagent is being added to the reaction and \( k \) is a constant.

a. [5 points] Write down the second-order linear equation which has \( r_1 \) as its solution.

b. [5 points] If \( f(t) = \cos(\omega t) \) is the dashed curve in the figure below, for what values of \( k \), if any, could the long-term behavior of \( r_1 \) be that shown by the solid curve? Explain your answer.

c. [5 points] If \( f(t) = A_0 \), a constant, for what values of \( k \), if any, could the phase portrait for this system be similar to that shown in the figure below? Explain your answer.
4. [13 points] Consider the RLC circuit shown to the right, below. This is modeled by \( y'' + ky' + 2y = g(t) \), where \( g(t) \) is the derivative of the input voltage and \( 0 < k < 2\sqrt{2} \) is proportional to the resistance of the resistor.

a. [9 points] If \( g(t) = 4 \cos(t) \), find the steady state response to the input. Write your answer in the form \( R \cos(t - \alpha) \).

b. [4 points] The amplitude of the steady state response to the forcing \( g(t) = 4 \cos(\omega t) \) is shown below, as a function of \( \omega \). What is the value of \( k \) in the equation? Why?
5. [14 points] For the first two of the following, identify each as true or false, by circling “True” or “False” as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer. For the last give a short answer explaining the indicated question.

a. [4 points] For some constant \( \omega \) and \( k \), a solution to the mechanical system \( y'' + 2y' + ky = \cos(\omega t) \) could be that shown to the right.

True  False

b. [4 points] Let \( F(s) = \frac{s^2 + 1}{s^2 + 3s + 5} \). There is some piecewise continuous function \( f(t) \), of exponential order, for which \( \mathcal{L}\{f(t)\} = F(s) \).

True  False

c. [6 points] Your friends Anna and Andrew are solving the two problems \( y'' + 0.1y' + y = 0, \ y(0) = 0, \ y'(0) = 1 \) and \( z'' + 0.1z' + z = 1(t - 3), \ z(0) = 0, \ z'(0) = 0 \). Anna thinks that \( z(t) = y(t - 3) \), while Andrew thinks they are different. Explain why they are both partly correct.
6. [14 points] Find solutions to each of the following, as indicated.

a. [7 points] Find the general solution to $y'' + y' = \frac{1}{1 + e^t}$. (Hint: $\int \frac{1}{1 + e^t} dt = t - \ln(1 + e^t)$.)

b. [7 points] Find the solution to $y'' - y' = u_2(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$
7. [14 points] Recall the linearized version of our laser model,
\begin{align*}
u' &= -\gamma(Au + v) \\
v' &= (A - 1)u.
\end{align*}
Consider the case of constant forcing \((A = \text{a constant})\) and the initial conditions \(u(0) = u_0, v(0) = 0\).

\textbf{a.} [6 points] Find a system of algebraic equations for \(U(s) = \mathcal{L}\{u(t)\}\) and \(V(s) = \mathcal{L}\{v(t)\}\).

\textbf{b.} [4 points] Solve your system from (a) to find \(U(s)\) and \(V(s)\).
\((If\ you\ are\ unable\ to\ solve\ (a),\ consider\ the\ system\ (s+a)U+bV = u_0, -cu+(s+a)V = 0.\))

\textbf{c.} [4 points] Recall that in the lab we solved the characteristic equation \(\lambda^2 + \gamma A \lambda + \gamma (A - 1) = 0\), finding \(\lambda^2 + \gamma A \lambda + \gamma (A - 1) = (\lambda + \frac{1}{2} \gamma A)^2 + (\frac{1}{4} \gamma^2 A^2 + \gamma (A - 1)) = (\lambda + \mu)^2 + \nu^2 = 0\) (so that \(\lambda = -\mu \pm i\nu\)). Use this to rewrite your solutions in (b) in terms of \(\mu\) and \(\nu\). Find \(u(t) = \mathcal{L}^{-1}\{U(s)\}\) and \(v(t) = \mathcal{L}^{-1}\{V(s)\}\) in terms of \(\mu\) and \(\nu\).
\((If\ you\ are\ stuck,\ assume\ the\ denominator\ of\ your\ U\ and\ V\ is\ of\ the\ form\ (s + a)^2 + b^2.\))
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