Math 216 — Final Exam

19 December, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Solve $2ty' + y = 5t^2$, $y(1) = 4$

b. [6 points] Find the general solution to $2y'' + y' + 2y = t\,e^{-t}$.

2. [12 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the general solution to the system $x' = x + 2y$, $y' = 6x + 2y$.

b. [6 points] Solve $y'' + 4y' + 4y = u_2(t)e^{-3(t-2)}$, $y(0) = 0$, $y'(0) = 7$.

3. [16 points] Consider a model for interacting populations x_1 and x_2 given by

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

a. [2 points] What type of interaction do you think there is between these populations (how does the interaction affect each population)? Explain.

b. [4 points] Find all critical points for this system.

c. [6 points] The Jacobian for this system is $J(x_1,x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each. *(This problem part continues on the next page.)*

Problem 3, continued. We are considering the system

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

c. *Continued: we are solving the problem*
The Jacobian for this system is $J(x_1, x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each.

d. [4 points] If the population of x_1 was initially large and that of x_2 small, sketch a qualitatively accurate graph of x_1 and x_2 as functions of time. What happens to the populations for large times?

4. [10 points] Consider the system

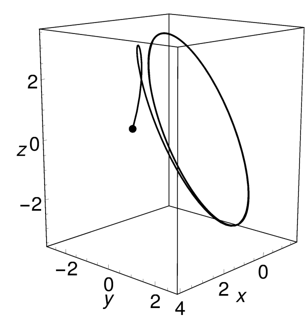
$$x' = -10x + 10y, \quad y' = r\,x - y - xz, \quad z' = 4z + xy.$$

For $r > 1$ this has a critical point $P = (2\sqrt{r-1}, 2\sqrt{r-1}, r-1)$. Let $\mathbf{A}$ be the matrix that gives the linearization of this system at P, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

a. [6 points] When $r = 34$, eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = -15$, $\lambda_2 = -13.3i$, and $\lambda_3 = 13.3i$, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0.4 - 0.3i \\ -0.8i \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0.4 + 0.3i \\ 0.8i \\ 1 \end{pmatrix}$.

Write a real-valued general solution to the linearization of the system in this case.

b. [4 points] Could the phase space trajectory shown to the right be that for the linearized system from (a)? Could it be that for the nonlinear system? Explain.



5. [12 points] Consider the nonlinear system

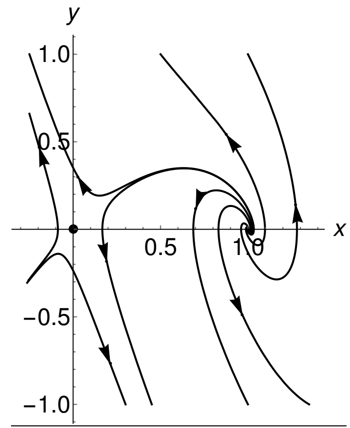
$$x' = y, \quad y' = -3x - 2y + r\,x^2.$$

Four possible phase portraits for this system are shown along the right side of the page.

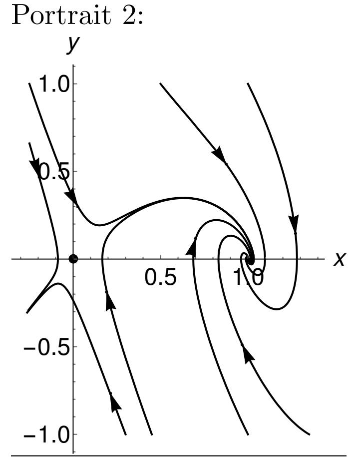
a. [4 points] If one of the graphs is to match this system, what is the value of the parameter r? Why?

b. [8 points] Given the value of r you found in **(a)**, which, if any, of the phase portraits could be that for this system? Why?

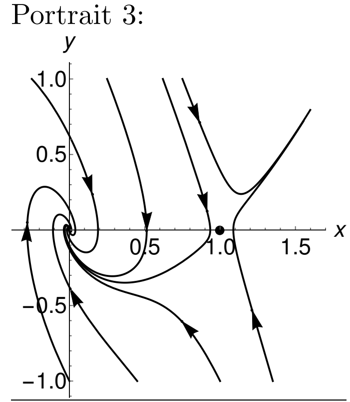
Portrait 1:



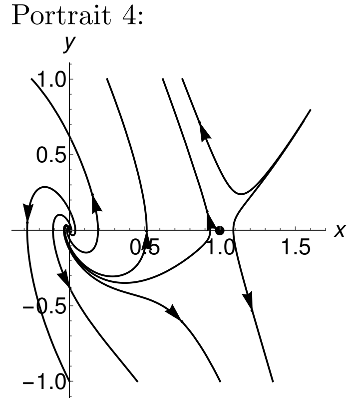
Portrait 2:



Portrait 3:



Portrait 4:



6. [12 points] For the following, identify each as true or false by circling "True" or "False" as appropriate. Then, if it is true, provide a short (one sentence) explanation indicating why it is true; if false, explain why or provide a counter-example.

a. [3 points] Let $\mathbf{A}$ be a 3×3 matrix with characteristic polynomial $p(\lambda) = \lambda^3 + 4\lambda^2 + \lambda - 6$. Then the origin is an asymptotically stable critical point of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

True False

b. [3 points] Consider the equation $y' = f(t, y)$, with f continuous for all values of t and y. We can solve this either by using an integrating factor or by separating variables (though in the latter case we may not be able to get an explicit solution for y).

True False

c. [3 points] While we cannot solve the nonlinear system $x' = x - x^2 - xy + \sin(t)$, $y' = y + xy$, we can obtain a good qualitative understanding of solutions by linearizing around critical points and sketching a phase portrait.

True False

d. [3 points] Long-term solutions to the system $y'' + 4y = 3\cos(4t)$ will be periodic.

True False

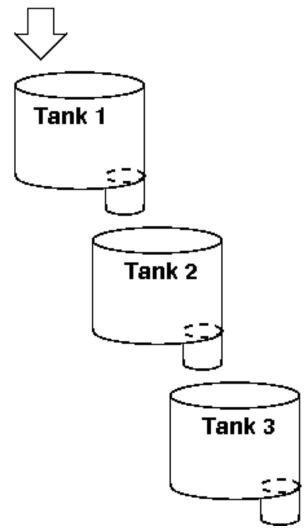
7. [12 points] Suppose that we are considering a system of two linear, constant-coefficient differential equations for x_1 and x_2 given in matrix form by $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$. We know that eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -3$.

a. [4 points] Suppose that $\mathbf{g} = \mathbf{0}$. If we rewrite the system as a single second-order linear equation in one of x_1 or x_2, what is the equation?

b. [4 points] Suppose now that $\mathbf{g}$ is nonzero, and let $x_1(0) = x_2(0) = 0$. If we apply the Laplace transform to the system and solve for $X_1 = \mathcal{L}\{x_1\}$ we will get $X_1 = G(s)/H(s)$, where $G(s)$ is a transform involving the components of $\mathbf{g}$. What is $H(s)$? Explain how you know.

c. [4 points] Finally suppose the eigenvectors of $\mathbf{A}$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and we solve the original system with the forcing term $\mathbf{g} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What is $\mathbf{x}_c$? What will the general solution look like? Specify all functions in your answer. (You may leave your solution in terms of $\mathbf{A}$ if you provide the matrix equation you would need to determine it completely.)

8. [7 points] Consider a three tank system as suggested by the figure to the right, below. The volumes of the three tanks are V_1, V_2 and V_3; suppose that they are initially full of water, and that there is x_0 kg of a contaminant in the first tank. Pure water is added to tank 1 at a rate of r liters/second, and the well-mixed mixture moves at the same rate from tank 1 to tank 2, from tank 2 to tank 3, and out of tank 3. Let x_1, x_2 and x_3 be the amount of the contaminant in tanks 1, 2, and 3. Write a system of differential equations, in matrix form, for x_1, x_2 and x_3. Indicate the initial condition that completes the initial value problem.



9. [7 points] Consider the mechanical system $y'' + ky' + 3y = g(t)$, $y(0) = 1$, $y'(0) = 3$. Find a $g(t)$ and all values of k for which of the following will be true:
(a) the steady state response of the system will be purely sinusoidal with period π;
(b) the response to the initial conditions will have halved in amplitude by the time $t = 5$; and
(c) the system is underdamped.

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- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta_c(t)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$