Math 216 — First Midterm

13 October, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find the general solution to $y' = \sin(t) - \dfrac{\sin(t)}{\cos(t)}\,y$.

Solution: This is linear and not separable, so we must use an integrating factor. The equation may be rewritten as

$$y' + \frac{\sin(t)}{\cos(t)}\,y = \sin(t),$$

so the integrating factor is $\mu = e^{\int \frac{\sin(t)}{\cos(t)}\,dt} = e^{-\ln(|\cos(t)|)} = 1/\cos(t)$ (we may drop the $\pm$ because of the constant of integration which will divide out from the equation after multiplication by μ). Multiplying through by this, we have

$$(\frac{1}{\cos(t)}\,y)' = \frac{\sin(t)}{\cos(t)},$$

so that after integrating both sides we have

$$\frac{1}{\cos(t)}\,y = -\ln(|\cos(t)|) + C,$$

and $y = -\cos(t)\ln(|\cos(t)|) + C\cos(t)$.

b. [7 points] Find a solution, explicit or implicit, for y, if

$$y' = \frac{1+\sin(t)}{1+\cos(y)}, \qquad y(\pi) = 0.$$

Solution: This is nonlinear, but separable. Separating variables, we have

$$(1+\cos(y))y' = 1+\sin(t).$$

Integrating, we have $y + \sin(y) = t - \cos(t) + C$. Applying the initial condition, we have $0 + 0 = \pi + 1 + C$, so $C = -1-\pi$, and our (implicit) solution is

$$y + \sin(y) = t - \cos(t) - (1+\pi).$$

2. [16 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [8 points] The general solution to the system $x_1' = 2x_1 + 3x_2$, $x_2' = x_1 + 4x_2$

Solution: This is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. We know solutions will be of the form $\mathbf{x} = \mathbf{v}\, e^{\lambda t}$, where $\mathbf{v}$ and λ are the eigenvectors and eigenvalues of $\mathbf{A}$. Solving for these, we require

$$\begin{aligned}\Delta = \det(\mathbf{A} - \lambda\mathbf{I}) = \det(\begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix} &= (\lambda-2)(\lambda-4) - 3 \\ &= \lambda^2 - 6\lambda + 5 = (\lambda-5)(\lambda-1) = 0.\end{aligned}$$

Thus $\lambda = 1$ or $\lambda = 5$. If $\lambda = 1$, the eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies $v_1 + 3v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Similarly, if $\lambda = 5$, the eigenvector satisfies $v_1 - v_2 = 0$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.$$

b. [8 points] The solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Solution: Again, we look for the eigenvalues and eigenvectors of the coefficient matrix. We have $\Delta = (1-\lambda)(-1-\lambda) + 2 = \lambda^2 + 1 = 0$, so that $\lambda = \pm i$. If $\lambda = i$, the eigenvector $\mathbf{v}$ satisfies

$$\begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that from the first row we see we may take $v_1 = 2$ and $v_2 = -1 + i$. Thus a complex-valued solution

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1+i \end{pmatrix} (\cos(t) + i\sin(t)) = \begin{pmatrix} 2\cos(t) + 2i\sin(t) \\ -\cos(t) - \sin(t) + i(\cos(t) - \sin(t)) \end{pmatrix}.$$

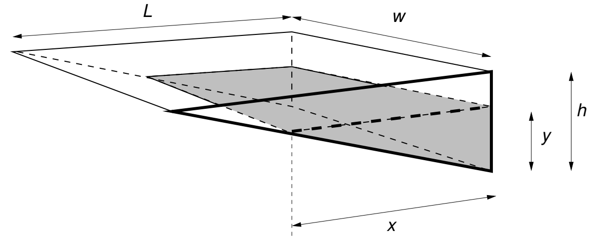
The real and imaginary parts of this are linearly independent solutions to the system, and so a real-valued general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) \\ -\cos(t) - \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}.$$

To get the desired initial condition, we take $c_1 = 2$ and $c_2 = 0$, so that

$$\mathbf{x} = \begin{pmatrix} 4\cos(t) \\ -2(\cos(t) + \sin(t)) \end{pmatrix}.$$

3. [12 points] Consider an abandoned zero-entry pool as suggested by the figure to the right, below. (The front face of the pool is shown with bold lines.) In the figure, w, L and h are the fixed dimensions of the pool, and x and y characterize the part that is filled with water. The corresponding volume of the filled section is $V = \frac{1}{2}\, w\, x\, y$.



a. [8 points] If the pool slowly evaporates at a volumetric rate proportional to its top surface area, write a differential equation for the volume of the water in the pool.

Solution: We have $\frac{dV}{dt} = -\alpha\, x\, w$. By similar triangles, $x/y = L/h$, so that $y = hx/L$. Thus we can relate x and V using the volume formula provided: $V = \frac{1}{2}\, w\, x\, y = \frac{hw}{2L}\, x^2$, and so $x = \sqrt{\frac{2L}{hw}}\,\sqrt{V}$. Thus

$$\frac{dV}{dt} = -\alpha\sqrt{\frac{2Lw}{h}}\sqrt{V} = -k\sqrt{V}.$$

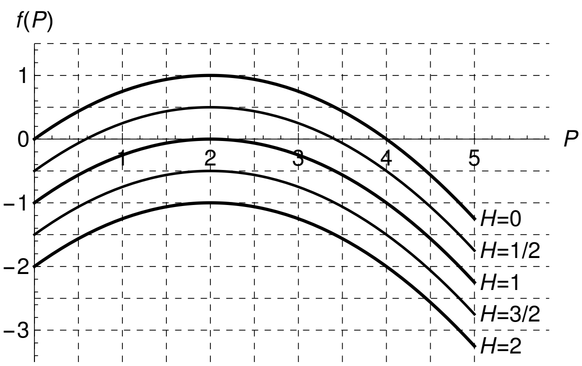
b. [4 points] Solve your equation from (**a**) with the initial condition $V(0) = V_0$. At what time is the pool finally empty? *(If you are unable to find an equation in (a), you may proceed with the equation $V' = -k\sqrt{V}$.)*

Solution: We solve by separating variables: $V^{-1/2}\, V' = -k$, so that on integrating both sides of the equation we have $2\sqrt{V} = -kt + C'$, so that $V = (C - kt/2)^2$. Applying the initial condition, $V(0) = C^2 = V_0$, so that $V = (\sqrt{V_0} - kt/2)^2$. The pool is empty when

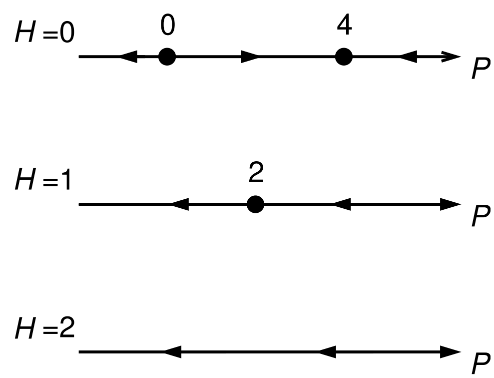
$$t = 2\frac{\sqrt{V_0}}{k} = 2\sqrt{\frac{V_0 hw}{L\alpha^2}}.$$

4. [18 points] A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.

a. [6 points] Plot phase lines for this equation when $H = 0$, $H = 1$ and $H = 2$. For each, identify all equilibrium solutions and their stability.

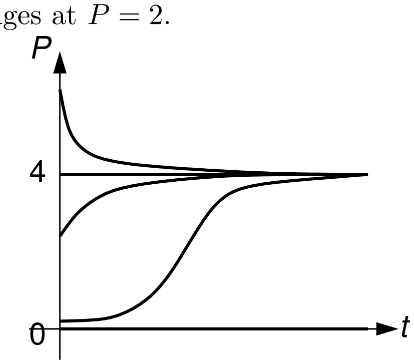


Solution: The phase lines are shown to the right. For $H = 0$, there are two equilibrium solutions, $P = 0$ and $P = 4$, with $P = 4$ being asymptotically stable and $P = 0$ unstable. For $H = 1$, there is a single equilibrium solution, $P = 2$, which is semi-stable (or, unstable). For $H = 2$ there are no equilibrium solutions, and all initial conditions strictly decrease. (Note that there is some ambiguity in what happens for $P < 0$, which is not shown in the graph and non-physical. Here we sketch the behaviors there by using the equation, which is defined for $P < 0$.)

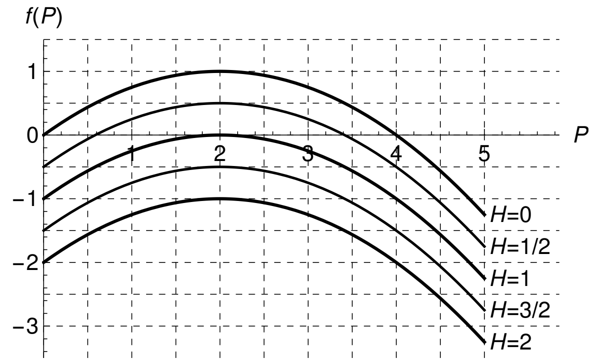


b. [5 points] Sketch qualitatively accurate solution curves for the case $H = 0$. Include enough initial conditions to show all solution behaviors.

Solution: Given the phase line above, we get the curves shown below. Note that the concavity of solutions changes at $P = 2$.

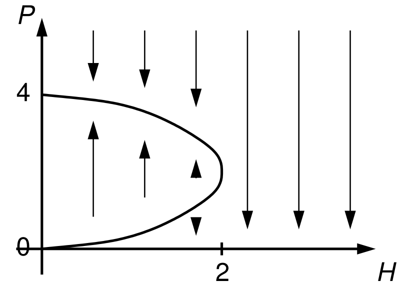


Problem 4, continued. Instructions are reproduced here:
A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where K is a limiting population and H the harvesting rate. P and K are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of K, the graphs of $f(P)$ are as in the graph shown below.



c. [4 points] This problem and your work on it provide an example of a model with a bifurcation. Draw the bifurcation diagram for this on the axes provided below.

Solution: The bifurcation diagram is below. The curve shown gives the two branches of a square root, with a base at $H = 2$.



d. [3 points] Explain what your work in the preceding indicates about the long-term survival of the harvested population (fish).

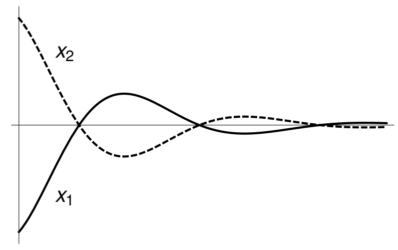
Solution: This indicates that if the harvesting is too high, the population will crash and go to zero. If it is low enough there is a stable larger population and things will continue as one might desire. The transition occurs at $H = 1$; below this, there is a stable "large" population of fish.

5. [16 points] Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{1}$$

for some real-valued, constant, 2×2 matrix $\mathbf{A}$. Suppose that one solution to (1) is $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$. Identify each of the following as true or false, by circling "True" or "False" as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer.

a. [4 points] A possible component plot of solutions to (1) is



Solution: This cannot be a correct plot, because the solutions x_1 and x_2 are decaying oscillatory solutions; given one solution with only real exponentials, we know that there can be no oscillatory solutions.

True **[False]**

b. [4 points] The general solution to (1) could be $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$.

[True] False

Solution: This could certainly be correct; we need only that the second eigenvalue of $\mathbf{A}$ be $\lambda = 1$ with corresponding eigenvector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (The reversed sign on the first solution is immaterial, as the eigenvectors are only unique up to a constant multiple.)

c. [4 points] The equation $\mathbf{A}\mathbf{w} = -\mathbf{w}$ has infinitely many solutions $\mathbf{w}$.

[True] False

Solution: Because we know one solution to the system is the $\mathbf{x}$ given, we know that one of the eigenvalues of $\mathbf{A}$ is $\lambda = -1$. This is just the eigenvalue problem with $\lambda = -1$ plugged in, so we know that there are an infinite number of solutions.

d. [4 points] An eigenvalue of the matrix $\mathbf{A}$ could be $\lambda = 1 + i$.

True **[False]**

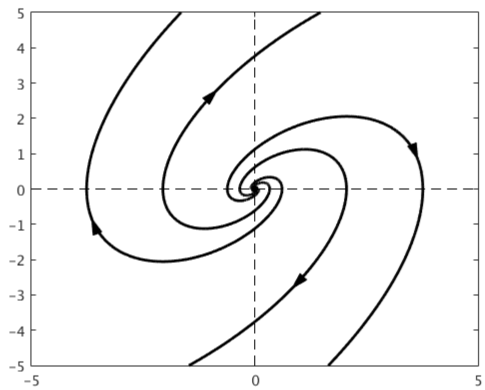
Solution: We know that $\lambda = -1$ is one eigenvalue, that there are at most two (because A is 2×2), and the complex eigenvalues must come as a complex-conjugate pair.

6. [12 points] Consider the system of equations

$$x_1' = x_2$$
$$x_2' = -x_1 + \alpha x_2,$$

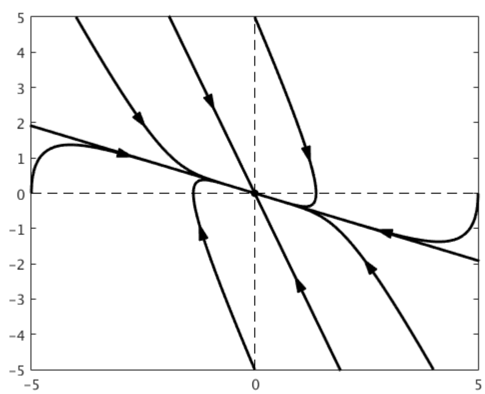
where α is a real-valued constant. For each of the phase portraits shown below, indicate all values for α that could result in this sytem having a phase portrait of that type and with the indicated stability. If it is not possible, write "**not possible**" and give a short explanation why.

a. [6 points]



Solution: For both of these we need the eigenvalues; here, they are given by $\lambda(\alpha-\lambda)+1 = 0$, or $\lambda^2 - \alpha\lambda - 1 = 0$, so $\lambda = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4}$. For this phase portrait we need a complex eigenvalue with positive real part, so $0 < \alpha < 2$.

b. [6 points]



Solution: Using the value of λ found above, we require that there be two real, negative roots. Thus we need $\alpha < -2$.

7. [12 points] The equation of motion for a damped, nonlinear pendulum is

$$\theta'' + c\,\theta' + k\,\sin(\theta) = 0,$$

where θ is the angle the pendulum makes with its midline, and c and k are positive (non-zero) real-valued constants.

a. [4 points] Rewrite this as a system of equations in $x = \theta$ and $y = \theta'$.

Solution: Making these substitutions, the equation becomes $y' = -k\,\sin(x) - cy$, and we have $x' = y$. Thus the system is

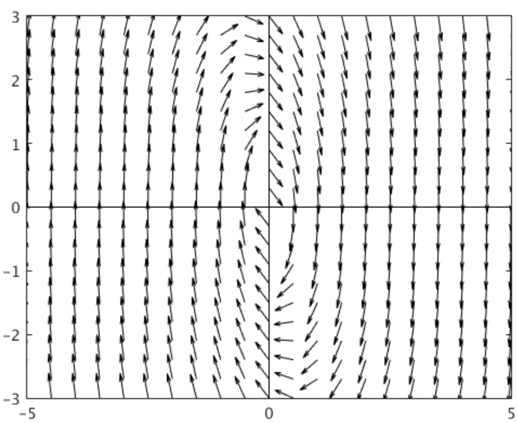
$$x' = y$$
$$y' = -k\,\sin(x) - cy.$$

b. [4 points] Find all critical points for your system from (a).

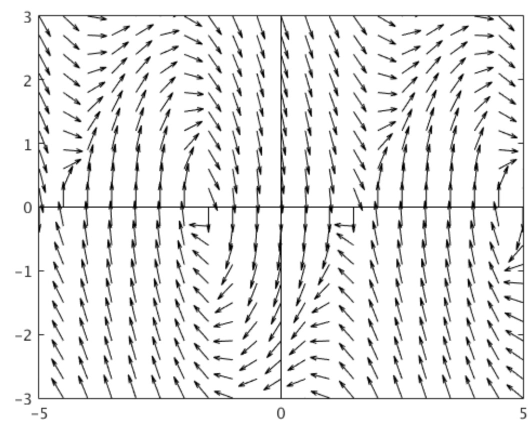
Solution: We require that $x' = 0$ and $y' = 0$, so that $y = 0$ and $x = \pm n\pi$, for n an integer.

c. [4 points] Which of the direction fields below could be that of your system in (a)? Why?

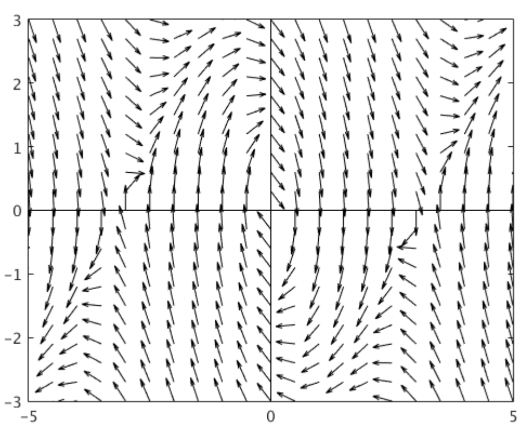
i.



ii.



iii.



Solution: First, note that there are multiple critical points for this system; thus (i), which shows only a critical point at $(0, 0)$, cannot be correct. Next, note that $(0, 0)$ is a critical point, which is not the case for (ii). Thus (iii) must be the correct direction field.