1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find the general solution to \( y' = \sin(t) - \frac{\sin(t)}{\cos(t)} y \).

*Solution:* This is linear and not separable, so we must use an integrating factor. The equation may be rewritten as

\[
y' + \frac{\sin(t)}{\cos(t)} y = \sin(t),
\]

so the integrating factor is \( \mu = e^{\frac{\sin(t)}{\cos(t)} dt} = e^{\frac{-\ln(|\cos(t)|)}{\cos(t)}} = 1/\cos(t) \) (we may drop the \( \pm \) because of the constant of integration which will divide out from the equation after multiplication by \( \mu \)). Multiplying through by this, we have

\[
\left( \frac{1}{\cos(t)} y \right)' = \frac{\sin(t)}{\cos(t)},
\]

so that after integrating both sides we have

\[
\frac{1}{\cos(t)} y = -\ln(|\cos(t)|) + C,
\]

and \( y = -\cos(t) \ln(|\cos(t)|) + C \cos(t) \).

b. [7 points] Find a solution, explicit or implicit, for \( y \), if

\[
y' = \frac{1 + \sin(t)}{1 + \cos(y)}, \quad y(\pi) = 0.
\]

*Solution:* This is nonlinear, but separable. Separating variables, we have

\[
(1 + \cos(y))y' = 1 + \sin(t).
\]

Integrating, we have \( y + \sin(y) = t - \cos(t) + C \). Applying the initial condition, we have

\( 0 + 0 = \pi + 1 + C \), so \( C = -1 - \pi \), and our (implicit) solution is

\[
y + \sin(y) = t - \cos(t) - (1 + \pi).
\]
2. [16 points] Find real-valued solutions to each of the following, as indicated. (Note that minimal partial credit will be given on this problem.)

   a. [8 points] The general solution to the system \( x'_1 = 2x_1 + 3x_2, \ x'_2 = x_1 + 4x_2 \)

      Solution: This is \( x' = Ax \) for \( A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \). We know solutions will be of the form \( x = v e^{\lambda t} \), where \( v \) and \( \lambda \) are the eigenvectors and eigenvalues of \( A \). Solving for these, we require
      \[
      \Delta = \det(A - \lambda I) = \det\begin{pmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{pmatrix} = (\lambda - 2)(\lambda - 4) - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0.
      \]
      Thus \( \lambda = 1 \) or \( \lambda = 5 \). If \( \lambda = 1 \), the eigenvector \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) satisfies \( v_1 + 3v_2 = 0 \), so that \( v = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \). Similarly, if \( \lambda = 5 \), the eigenvector satisfies \( v_1 - v_2 = 0 \), and \( v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Thus the general solution is
      \[
      x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}.
      \]

   b. [8 points] The solution to \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \) with \( \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \).

      Solution: Again, we look for the eigenvalues and eigenvectors of the coefficient matrix. We have \( \Delta = (1 - \lambda)(-1 - \lambda) + 2 = \lambda^2 + 1 = 0 \), so that \( \lambda = \pm i \). If \( \lambda = i \), the eigenvector \( v \) satisfies
      \[
      \begin{pmatrix} 1 - i & 2 \\ -1 & -1 + i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
      \]
      so that from the first row we see we may take \( v_1 = 2 \) and \( v_2 = -1 + i \). Thus a complex-valued solution
      \[
      x = \begin{pmatrix} 2 \\ -1 + i \end{pmatrix} (\cos(t) + i \sin(t)) = \begin{pmatrix} 2 \cos(t) + 2i \sin(t) \\ -\cos(t) - \sin(t) + i(\cos(t) - \sin(t)) \end{pmatrix}.
      \]
      The real and imaginary parts of this are linearly independent solutions to the system, and so a real-valued general solution is
      \[
      x = c_1 \begin{pmatrix} 2 \cos(t) \\ -\cos(t) - \sin(t) \end{pmatrix} + c_2 \begin{pmatrix} 2 \sin(t) \\ \cos(t) - \sin(t) \end{pmatrix}.
      \]
      To get the desired initial condition, we take \( c_1 = 2 \) and \( c_2 = 0 \), so that
      \[
      x = \begin{pmatrix} 4 \cos(t) \\ -2(\cos(t) + \sin(t)) \end{pmatrix}.
3. [12 points] Consider an abandoned zero-entry pool as suggested by the figure to the right, below. (The front face of the pool is shown with bold lines.) In the figure, $w$, $L$ and $h$ are the fixed dimensions of the pool, and $x$ and $y$ characterize the part that is filled with water. The corresponding volume of the filled section is $V = \frac{1}{2} w x y$.

a. [8 points] If the pool slowly evaporates at a volumetric rate proportional to its top surface area, write a differential equation for the volume of the water in the pool.

Solution: We have $\frac{dV}{dt} = -\alpha x w$. By similar triangles, $x/y = L/h$, so that $y = hx/L$. Thus we can relate $x$ and $V$ using the volume formula provided: $V = \frac{1}{2} w x y = \frac{hw}{2L} x^2$, and so $x = \sqrt{\frac{2L}{h} x^2}$. Thus

$$\frac{dV}{dt} = -\alpha \sqrt{\frac{2Lw}{h} \sqrt{V}} = -k\sqrt{V}.$$ 

b. [4 points] Solve your equation from (a) with the initial condition $V(0) = V_0$. At what time is the pool finally empty? (If you are unable to find an equation in (a), you may proceed with the equation $V' = -k\sqrt{V}$.)

Solution: We solve by separating variables: $V^{-1/2} V' = -k$, so that on integrating both sides of the equation we have $2\sqrt{V} = -kt + C'$, so that $V = (C - kt/2)^2$. Applying the initial condition, $V(0) = C^2 = V_0$, so that $V = (\sqrt{V_0} - kt/2)^2$. The pool is empty when

$$t = 2\sqrt{\frac{V_0}{k}} = 2\sqrt{\frac{V_0 h w}{L\alpha^2}}.$$
4. [18 points] A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where $K$ is a limiting population and $H$ the harvesting rate. $P$ and $K$ are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of $K$, the graphs of $f(P)$ are as in the graph shown below.

a. [6 points] Plot phase lines for this equation when $H = 0$, $H = 1$ and $H = 2$. For each, identify all equilibrium solutions and their stability.

Solution: The phase lines are shown to the right. For $H = 0$, there are two equilibrium solutions, $P = 0$ and $P = 4$, with $P = 4$ being asymptotically stable and $P = 0$ unstable. For $H = 1$, there is a single equilibrium solution, $P = 2$, which is semi-stable (or, unstable). For $H = 2$ there are no equilibrium solutions, and all initial conditions strictly decrease. (Note that there is some ambiguity in what happens for $P < 0$, which is not shown in the graph and non-physical. Here we sketch the behaviors there by using the equation, which is defined for $P < 0$.)

b. [5 points] Sketch qualitatively accurate solution curves for the case $H = 0$. Include enough initial conditions to show all solution behaviors.

Solution: Given the phase line above, we get the curves shown below. Note that the concavity of solutions changes at $P = 2$. 

[Diagram showing phase lines and solution curves for different values of $H$.]
**Problem 4, continued. Instructions are reproduced here:**

A model for a population with harvesting (e.g., a population of fish from which fish are caught) is $P' = f(P) = P(1 - \frac{P}{K}) - H$, where $K$ is a limiting population and $H$ the harvesting rate. $P$ and $K$ are measured in some unit—perhaps millions of pounds of fish. Suppose that for some value of $K$, the graphs of $f(P)$ are as in the graph shown below.

![Graph of f(P) with different harvesting rates H]

**c. [4 points]** This problem and your work on it provide an example of a model with a bifurcation. Draw the bifurcation diagram for this on the axes provided below.

**Solution:** The bifurcation diagram is below. The curve shown gives the two branches of a square root, with a base at $H = 2$.

![Bifurcation diagram]

**d. [3 points]** Explain what your work in the preceding indicates about the long-term survival of the harvested population (fish).

**Solution:** This indicates that if the harvesting is too high, the population will crash and go to zero. If it is low enough there is a stable larger population and things will continue as one might desire. The transition occurs at $H = 1$; below this, there is a stable “large” population of fish.
5. [16 points] Consider the system

\[ x' = Ax, \]  

for some real-valued, constant, 2 \times 2 matrix \( A \). Suppose that one solution to (1) is \( x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \). Identify each of the following as true or false, by circling “True” or “False” as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer.

a. [4 points] A possible component plot of solutions to (1) is

\[ x_1 \quad x_2 \]

\[ x_1 \quad x_2 \]

Solution: This cannot be a correct plot, because the solutions \( x_1 \) and \( x_2 \) are decaying oscillatory solutions; given one solution with only real exponentials, we know that there can be no oscillatory solutions.

True \hspace{1cm} False

b. [4 points] The general solution to (1) could be

\[ x = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t. \]

Solution: This could certainly be correct; we need only that the second eigenvalue of \( A \) be \( \lambda = 1 \) with corresponding eigenvector \( v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). (The reversed sign on the first solution is immaterial, as the eigenvectors are only unique up to a constant multiple.)

True \hspace{1cm} False

c. [4 points] The equation \( Aw = -w \) has infinitely many solutions \( w \).

Solution: Because we know one solution to the system is the \( x \) given, we know that one of the eigenvalues of \( A \) is \( \lambda = -1 \). This is just the eigenvalue problem with \( \lambda = -1 \) plugged in, so we know that there are an infinite number of solutions.

True \hspace{1cm} False

d. [4 points] An eigenvalue of the matrix \( A \) could be \( \lambda = 1 + i \).

Solution: We know that \( \lambda = -1 \) is one eigenvalue, that there are at most two (because \( A \) is 2 \times 2 \), and the complex eigenvalues must come as a complex-conjugate pair.

True \hspace{1cm} False
6. [12 points] Consider the system of equations

\[
\begin{align*}
  x'_1 &= x_2 \\
  x'_2 &= -x_1 + \alpha x_2,
\end{align*}
\]

where \( \alpha \) is a real-valued constant. For each of the phase portraits shown below, indicate all values for \( \alpha \) that could result in this system having a phase portrait of that type and with the indicated stability. If it is not possible, write “not possible” and give a short explanation why.

a. [6 points]

[Phase portrait image]

**Solution:** For both of these we need the eigenvalues; here, they are given by \( \lambda(\alpha - \lambda) + 1 = 0 \), or \( \lambda^2 - \alpha \lambda - 1 = 0 \), so \( \lambda = \frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha^2 - 4} \). For this phase portrait we need a complex eigenvalue with positive real part, so \( 0 < \alpha < 2 \).

b. [6 points]

[Phase portrait image]

**Solution:** Using the value of \( \lambda \) found above, we require that there be two real, negative roots. Thus we need \( \alpha < -2 \).
7. [12 points] The equation of motion for a damped, nonlinear pendulum is

\[ \theta'' + c \theta' + k \sin(\theta) = 0, \]

where \( \theta \) is the angle the pendulum makes with its midline, and \( c \) and \( k \) are positive (non-zero) real-valued constants.

a. [4 points] Rewrite this as a system of equations in \( x = \theta \) and \( y = \theta' \).

Solution: Making these substitutions, the equation becomes \( y' = -k \sin(x) - cy \), and we have \( x' = y \). Thus the system is

\[
\begin{align*}
    x' &= y \\
    y' &= -k \sin(x) - cy.
\end{align*}
\]

b. [4 points] Find all critical points for your system from (a).

Solution: We require that \( x' = 0 \) and \( y' = 0 \), so that \( y = 0 \) and \( x = \pm n\pi \), for \( n \) an integer.

c. [4 points] Which of the direction fields below could be that of your system in (a)? Why?

Solution: First, note that there are multiple critical points for this system; thus (i), which shows only a critical point at \((0, 0)\), cannot be correct. Next, note that \((0, 0)\) is a critical point, which is not the case for (ii). Thus (iii) must be the correct direction field.