Math 216 — Second Midterm
17 November, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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Some Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$</td>
</tr>
<tr>
<td>$\delta(t - c)$</td>
<td>$e^{-cs}$</td>
</tr>
</tbody>
</table>

A. $f'(t)$ | $sF(s) - f(0)$ |
A.1 $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ |
A.2 $f^{(n)}(t)$ | $s^nF(s) - \cdots - f^{(n-1)}(0)$ |

B. $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ |
C. $\ e^{ct} f(t)$ | $F(s-c)$ |
D. $u_c(t) f(t - c)$ | $e^{-cs} F(s)$ |
E. $f(t)$ (periodic with period $T$) | $\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) \, dt$ |
1. [16 points] Find real-valued solutions for each of the following, as indicated. (Note that minimal partial credit will be given on this problem.)

a. [8 points] Find the general solution to $y'' + 4y' + 13y = 26t$.

**Solution:** The general solution will be $y = y_c + y_p$, where $y_c$ solves the complementary homogenous problem and $y_p$ is a particular solutions. For $y_c$ we guess $y = e^{\lambda t}$, so that $\lambda^2 + 4\lambda + 13 = (\lambda + 2)^2 + 9 = 0$, and $\lambda = -2 \pm 3i$. Thus $y_c = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t)$.

For $y_p$ we use the method of undetermined coefficients, taking $y_p = At + B$. Plugging in, we have $0 + 8A + 13At + 13B = 26t$, so that $A = 2$ and $B = -8/13$. Thus the general solution is

$$y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t) + 2t - \frac{8}{13}.$$

b. [8 points] Solve $y'' + 6y' + 8y = e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

**Solution:** Again, with $y = e^{\lambda t}$, we have $\lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0$, so that $y_c = c_1 e^{-2t} + c_2 e^{-4t}$. Our guess for $y_p$ is then $y_p = Ate^{-2t}$ (where we have to multiply by $t$ because the forcing appears in the complementary homogeneous solution). The derivatives of $y_p$ are $y_p' = Ae^{-2t} - 2Ate^{-2t}$ and $y_p'' = -4Ae^{-2t} + 4Ate^{-2t}$. Plugging in, we have

$$(-4Ae^{-2t} + 4Ate^{-2t}) + (6Ae^{-2t} - 12Ate^{-2t}) + 8Ate^{-2t} = 2Ae^{-2t} = e^{-2t},$$

so that $A = \frac{1}{2}$. Thus the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{1}{2} t e^{-2t}.$$

Plugging in the initial conditions, $y(0) = c_1 + c_2 = 0$, and $y'(0) = -2c_1 - 4c_2 + \frac{1}{2} = 0$. Thus $c_1 = -c_2$, so that $c_2 = \frac{1}{4}$ and $c_1 = -\frac{1}{4}$, and $y = -\frac{1}{4} e^{-2t} + \frac{1}{4} e^{-4t} + \frac{1}{2} t e^{-2t}$. 

2. [14 points] Find each of the following. (Note that minimal partial credit will be given on this problem.)

a. [7 points] \( \mathcal{L}\{f(t)\} \), if \( f(t) = \begin{cases} 1 - t, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \)

**Solution:** Note that \( f(t) = (1-t)(1-u_1(t)) = -(t-1) + (t-1)u_1(t) \). From our table of transforms, we note that \( \mathcal{L}\{t\} = \frac{1}{s^2} \) and \( \mathcal{L}\{(t-1)\} = \frac{1}{s^2} - \frac{1}{s} \). Thus

\[
\mathcal{L}\{f(t)\} = -\left(\frac{1}{s^2} - \frac{1}{s}\right) + \frac{1}{s^2} e^{-s} = \frac{1}{s} + \frac{1}{s^2}(e^{-s} - 1).
\]

Alternately, we can find this by direct integration:

\[
\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} \, dt = \int_0^1 (1-t)e^{-st} \, dt = -\frac{1}{s} e^{-st} \bigg|_{t=1}^{t=0} - \int_0^1 t e^{-st} \, dt.
\]

The remaining integral we complete by parts with \( u = t \) and \( v' = e^{-st} \), so that \( \int t e^{-st} \, dt = -\frac{1}{s} t e^{-st} + \int \frac{1}{s} e^{-st} \, dt = -\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} \). Thus

\[
\mathcal{L}\{f(t)\} = \left(-\frac{1}{s} e^{-st} + \frac{1}{s} t e^{-st} + \frac{1}{s^2} e^{-st}\right) \bigg|_{t=1}^{t=0} = \frac{1}{s} + \frac{1}{s^2}(e^{-s} - 1).
\]

b. [7 points] \( Y(s) = \mathcal{L}\{y(t)\} \), if \( y'' + 9y = u_\pi(t) \cos(4(t - \pi)) \), \( y(0) = 1 \), \( y'(0) = 2 \).

**Solution:** Transforming, we have \( \mathcal{L}\{y'' + 9y\} = \mathcal{L}\{u_\pi(t) \cos(4(t - \pi))\} \), so that

\[
s^2 Y - s - 2 + 9Y = \frac{se^{-\pi s}}{s^2 + 16},
\]

and

\[
Y = \frac{s + 2}{s^2 + 9} + \frac{se^{-\pi s}}{(s^2 + 9)(s^2 + 16)}.
\]
3. [15 points] A chemical reaction with two reagents (chemicals) in amounts $r_1$ and $r_2$ that may be converted from one to the other may be modeled the system of first-order differential equations

\[
\begin{align*}
    r_1' &= -3r_1 + 9r_2 \\
    r_2' &= kr_1 - r_2 + f(t),
\end{align*}
\]

where $f(t)$ is the rate at which the second reagent is being added to the reaction and $k$ is a constant.

a. [5 points] Write down the second-order linear equation which has $r_1$ as its solution.

**Solution:** From the first equation in the system, we have $r_2 = \frac{1}{9} r_1' + \frac{1}{3} r_1$, so that $r_2' = \frac{1}{9} r''_1 + \frac{1}{3} r'_1$. Plugging into the second equation, we have

\[
\frac{1}{9} r''_1 + \frac{1}{3} r'_1 = k r_1 - \frac{1}{9} r'_1 - \frac{1}{3} r_1 + f(t), \quad \text{or} \quad r''_1 + 4r'_1 + (3 - 9k)r_1 = f(t).
\]

b. [5 points] If $f(t) = \cos(\omega t)$ is the dashed curve in the figure below, for what values of $k$, if any, could the long-term behavior of $r_1$ be that shown by the solid curve? Explain your answer.

**Solution:** The solution for $r_1$ will be of the form $r_1 = r_c + r_p$, where $r_p = R \cos(\omega t - \delta)$. Thus, for this to be the long-term solution we must have $r_c \to 0$. This will be the case when the roots of the characteristic polynomial have negative real parts. The characteristic equation is $\lambda^2 + 4\lambda + (3 - 9k) = (\lambda + 2)^2 + (-1 - 9k) = 0$, for which $\lambda = -2 \pm \sqrt{1 + 9k}$. Thus we need $1 + 9k < 4$, or $k < 1/3$, for the real part of $\lambda$ to be negative.

c. [5 points] If $f(t) = A_0$, a constant, for what values of $k$, if any, could the phase portrait for this system be similar to that shown in the figure below? Explain your answer.

**Solution:** Here we need the characteristic equation to have two negative, real roots. Thus we need $0 < 1 + 9k < 4$, so $-\frac{1}{9} < k < \frac{1}{3}$. 

4. [13 points] Consider the RLC circuit shown to the right, below. This is modeled by $y'' + ky' + 2y = g(t)$, where $g(t)$ is the derivative of the input voltage and $0 < k < 2\sqrt{2}$ is proportional to the resistance of the resistor.

a. [9 points] If $g(t) = 4 \cos(t)$, find the steady state response to the input. Write your answer in the form $R \cos(t - \alpha)$.

Solution: The steady state response will be the particular solution to the problem. Using the method of undetermined coefficients, let $y_p = A \cos t + B \sin t$. Then, plugging in and collecting terms in $\cos t$ and $\sin t$, we have

$$-A + Bk + 2A = 4, \quad \text{and} \quad -B - Ak + 2B = 0.$$ 

These are $A + Bk = 4$, $-Ak + B = 0$. Solving by taking $k$ times the first and adding, we have $(k^2 + 1)B = 4k$, so that $B = \frac{4k}{k^2 + 1}$. Then $A = \frac{4}{k^2 + 1}$. These are both positive, so in phase amplitude form we have

$$y_p = \sqrt{A^2 + B^2} \cos(t - \arctan(B/A)) = \frac{4}{\sqrt{k^2 + 1}} \cos(t - \arctan(k)).$$

b. [4 points] The amplitude of the steady state response to the forcing $g(t) = 4 \cos(\omega t)$ is shown below, as a function of $\omega$. What is the value of $k$ in the equation? Why?

Solution: The expression for $R$ above is for $\omega = 1$, so $\frac{4}{\sqrt{k^2 + 1}} = 2$, where we have read the value for $R$ from the figure. This gives $\sqrt{k^2 + 1} = 2$, so $k^2 + 1 = 4$ and $k = \pm \sqrt{3}$. Given that it is an RLC circuit we can discard the negative value, taking $k = \sqrt{3}$.
5. [14 points] For the first two of the following, identify each as true or false, by circling “True” or “False” as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer. For the last give a short answer explaining the indicated question.

**a. [4 points]** For some constant $\omega$ and $k$, a solution to the mechanical system $y'' + 2y' + ky = \cos(\omega t)$ could be that shown to the right.

**Solution:** This cannot be true; the forcing requires that the steady state solution be sinusoidal, and centered on the $t$-axis.

**b. [4 points]** Let $F(s) = \frac{s^2+1}{s^2+3s+5}$. There is some piecewise continuous function $f(t)$, of exponential order, for which $\mathcal{L}\{f(t)\} = F(s)$.

**Solution:** This is false, because $F(s) \rightarrow 1 \neq 0$ as $s \rightarrow \infty$. We know that all transforms of regular functions must go to zero as $s \rightarrow \infty$.

**c. [6 points]** Your friends Anna and Andrew are solving the two problems $y'' + 0.1y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$ and $z'' + 0.1z' + z = \delta(t-3), \quad z(0) = 0, \quad z'(0) = 0$. Anna thinks that $z(t) = y(t-3)$, while Andrew thinks they are different. Explain why they are both partly correct.

**Solution:** Note that the transforms of these problems give $Y = \frac{1}{s^2 + 0.1s + 1}$ and $Z = e^{-3s}/(s^2 + 0.1s + 1)$. Thus we know that $z(t) = y(t-3)u_3(t)$. The two are the same, with the ambiguity of the value of the derivative at $t = 3$—because $z$ has the step function there the value of $z'$ at $t = 3$ is not uniquely determined.
6. [14 points] Find solutions to each of the following, as indicated.

a. [7 points] Find the general solution to $y'' + y' = \frac{1}{1 + e^t}$. (Hint: $\int \frac{1}{1+e^t} \, dt = t - \ln(1 + e^t).$

Solution: Solutions to the homogeneous problem are $y_1 = 1$ and $y_2 = e^{-t}$. We cannot here use the method of undetermined coefficients, and so use variation of parameters instead. Let $y_p = u_1 + u_2 e^{-t}$. Then

$$u_1' + u_2' e^{-t} = 0 \quad \text{and} \quad -u_2' e^{-t} = \frac{1}{1 + e^t}.$$ 

The second equation gives $u_2' = -e^t (1 + e^t)^{-1}$, so that $u_2 = -\ln(1 + e^t)$. The first then gives $u_1' = -u_2' e^{-t} = 1/(1 + e^t)$. Following the hint, $u_1 = t - \ln(1 + e^t)$. Thus the general solution is

$$y = c_1 + c_2 e^{-t} + (t - \ln(1 + e^t)) - e^{-t} \ln(1 + e^t).$$

b. [7 points] Find the solution to $y'' - y' = u_2(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$.

Solution: We use Laplace transforms: transforming both sides of the equation, we have

$$s^2 Y - sY = \frac{1}{s}(e^{-2s} - e^{-3s}), \quad \text{so} \quad Y = \frac{1}{s^2(s-1)}(e^{-2s} - e^{-3s}).$$

To find the inverse transform of this, we first apply partial fractions to the rational function. Letting $\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$, we have after clearing the denominator

$$1 = As(s-1) + B(s-1) + Cs^2.$$ 

Setting $s = 0$, we have $B = -1$. If $s = 1$ we get $C = 1$. Finally, with $s = -1$, $1 = 2A + 2 + 1$, so that $A = -1$. Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -1 - t + e^t = f(t).$$

Thus, including the exponentials to get the expected step functions, we have

$$y = \mathcal{L}^{-1}\{Y\} = f(t-2)u_2(t) - f(t-3)u_3(t).$$
7. [14 points] Recall the linearized version of our laser model,

\[ u' = -\gamma(Au + v) \]
\[ v' = (A - 1)u. \]

Consider the case of constant forcing \((A = \text{a constant})\) and the initial conditions \(u(0) = u_0, v(0) = 0\).

(a) [6 points] Find a system of algebraic equations for \(U(s) = \mathcal{L}\{u(t)\}\) and \(V(s) = \mathcal{L}\{v(t)\}\).

**Solution:** Transforming both sides of the system, we have

\[ sU - u_0 = -\gamma AU - \gamma V \]
\[ sV - 0 = (A - 1)U, \]

or

\[ (s + \gamma A)U + \gamma V = u_0 \]
\[ -(A - 1)U + sV = 0. \]

(b) [4 points] Solve your system from (a) to find \(U(s)\) and \(V(s)\).

*If you are unable to solve (a), consider the system \((s+a)U + bV = u_0, -cU + (s+a)V = 0.\)*

**Solution:** Note that we have \(V = \frac{A-1}{s} U\) from the second equation. Plugging this into the first, we have

\[ (s + \gamma A)U + \frac{\gamma(A - 1)}{s} U = \frac{s^2 + \gamma As + \gamma(A - 1)}{s} U = u_0, \]

so that \(U = \frac{s u_0}{s^2 + \gamma As + \gamma(A - 1)}\). Plugging this in for \(V\), we have \(V = \frac{(A-1)u_0}{s^2 + \gamma As + \gamma(A - 1)}\).

With the alternate system, we have \(U = \frac{s + a}{c} V\), so that \(((s + a)^2 + bc)V = u_0 c,\) or \(V = \frac{u_0 c}{(s+a)^2 + bc}\), and \(U = \frac{u_0 (s+a)}{(s+a)^2 + bc}\).

(c) [4 points] Recall that in the lab we solved the characteristic equation \(\lambda^2 + \gamma A\lambda + \gamma(A - 1) = 0\), finding \(\lambda^2 + \gamma A\lambda + \gamma(A - 1) = (\lambda + \frac{1}{2} \lambda A)^2 + (-\frac{1}{4} \gamma^2 A^2 + \gamma(A - 1)) = (\lambda + \mu)^2 + \nu^2 = 0\) (so that \(\lambda = -\mu \pm i\nu\)). Use this to rewrite your solutions in (b) in terms of \(\mu\) and \(\nu\). Find \(u(t) = \mathcal{L}^{-1}\{U(s)\}\) and \(v(t) = \mathcal{L}^{-1}\{V(s)\}\) in terms of \(\mu\) and \(\nu\).

*If you are stuck, assume the denominator of your \(U\) and \(V\) is of the form \((s + a)^2 + b^2\).*

**Solution:** We have

\[ U = \frac{s u_0}{(s + \mu)^2 + \nu^2} = \frac{(s + \mu) u_0}{(s + \mu)^2 + \nu^2} - \frac{\mu u_0}{(s + \mu)^2 + \nu^2}, \]
\[ V = \frac{(A-1)u_0}{(s + \mu)^2 + \nu^2}. \]

Inverting these, we have

\[ u(t) = u_0 e^{-\mu t} \cos(\nu t) - \frac{u_0}{\nu} e^{-\mu t} \sin(\nu t), \]
\[ v(t) = \frac{(A-1)}{\nu} u_0 e^{-\mu t} \sin(\nu t). \]