Math 216 — Second Midterm

17 November, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$

1. [16 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [8 points] Find the general solution to $y'' + 4y' + 13y = 26t$.

Solution: The general solution will be $y = y_c + y_p$, where y_c solves the complementary homogenous problem and y_p is a particular solutions. For y_c we guess $y = e^{\lambda t}$, so that $\lambda^2+4\lambda+13 = (\lambda+2)^2+9 = 0$, and $\lambda = -2\pm 3i$. Thus $y_c = c_1 e^{-2t}\cos(3t)+c_2 e^{-2t}\sin(3t)$. For y_p we use the method of undetermined coefficients, taking $y_p = At + B$. Plugging in, we have

$$0 + 8A + 13At + 13B = 26t,$$

so that $A = 2$ and $B = -8/13$. Thus the general solution is

$$y = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + 2t - \frac{8}{13}.$$

b. [8 points] Solve $y'' + 6y' + 8y = e^{-2t}$, $y(0) = 0$, $y'(0) = 0$.

Solution: Again, with $y = e^{\lambda t}$, we have $\lambda^2 + 6\lambda + 8 = (\lambda + 2)(\lambda + 4) = 0$, so that $y_c = c_1 e^{-2t} + c_2 e^{-4t}$. Our guess for y_p is then $y_p = Ate^{-2t}$ (where we have to multiply by t because the forcing appears in the complementary homogeneous solution). The derivatives of y_p are $y_p' = Ae^{-2t} - 2Ate^{-2t}$ and $y_p'' = -4Ae^{-2t} + 4Ate^{-2t}$. Plugging in, we have

$$(-4Ae^{-2t} + 4Ate^{-2t}) + (6Ae^{-2t} - 12Ate^{-2t}) + 8Ate^{-2t} = 2Ae^{-2t} = e^{-2t},$$

so that $A = \frac{1}{2}$. Thus the general solution is

$$y = c_1 e^{-2t} + c_2 e^{-4t} + \frac{1}{2}\, t\, e^{-2t}.$$

Plugging in the initial conditions, $y(0) = c_1 + c_2 = 0$, and $y'(0) = -2c_1 - 4c_2 + \frac{1}{2} = 0$. Thus $c_1 = -c_2$, so that $c_2 = \frac{1}{4}$ and $c_1 = -\frac{1}{4}$, and $y = -\frac{1}{4}\, e^{-2t} + \frac{1}{4}\, e^{-4t} + \frac{1}{2}\, t\, e^{-2t}$.

2. [14 points] Find each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $\mathcal{L}\{f(t)\}$, if $f(t) = \begin{cases} 1-t, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$

Solution: Note that $f(t) = (1-t)(1-u_1(t)) = -(t-1) + (t-1)u_1(t)$. From our table of transforms, we note that $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{(t-1)\} = \frac{1}{s^2} - \frac{1}{s}$. Thus

$$\mathcal{L}\{f(t)\} = -\left(\frac{1}{s^2} - \frac{1}{s}\right) + \frac{1}{s^2}\,e^{-s} = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right).$$

Alternately, we can find this by direct integration:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^1 (1-t)e^{-st}\,dt = -\frac{1}{s}\,e^{-st}\Big|_{t=0}^{t=1} - \int_0^1 t\,e^{-st}\,dt.$$

The remaining integral we complete by parts with $u = t$ and $v' = e^{-st}$, so that $\int t\,e^{-st}\,dt = -\frac{1}{s}\,t\,e^{-st} + \int \frac{1}{s}\,e^{-st}\,dt = -\frac{1}{s}\,t\,e^{-st} - \frac{1}{s^2}\,e^{-st}$. Thus

$$\mathcal{L}\{f(t)\} = \left(-\frac{1}{s}\,e^{-st} + \frac{1}{s}\,t\,e^{-st} + \frac{1}{s^2}\,e^{-st}\right)\Bigg|_{t=0}^{t=1} = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s} - 1\right).$$

b. [7 points] $Y(s) = \mathcal{L}\{y(t)\}$, if $y'' + 9y = u_\pi(t)\cos(4(t-\pi))$, $y(0) = 1$, $y'(0) = 2$.

Solution: Transforming, we have $\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{u_\pi(t)\cos(4(t-\pi))\}$, so that

$$s^2\,Y - s - 2 + 9Y = \frac{s\,e^{-\pi s}}{s^2 + 16},$$

and

$$Y = \frac{s+2}{s^2+9} + \frac{s\,e^{-\pi s}}{(s^2+9)(s^2+16)}.$$

3. [15 points] A chemical reaction with two reagents (chemicals) in amounts r_1 and r_2 that may be converted from one to the other may be modeled the system of first-order differential equations

$$r_1' = -3r_1 + 9r_2$$
$$r_2' = k\,r_1 - r_2 + f(t),$$

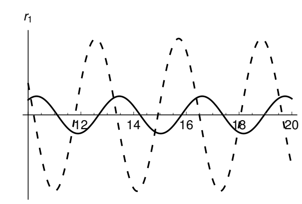
where $f(t)$ is the rate at which the second reagent is being added to the reaction and k is a constant.

a. [5 points] Write down the second-order linear equation which has r_1 as its solution.

Solution: From the first equation in the system, we have $r_2 = \frac{1}{9}\,r_1' + \frac{1}{3}\,r_1$, so that $r_2' = \frac{1}{9}\,r_1'' + \frac{1}{3}\,r_1'$. Plugging into the second equation, we have

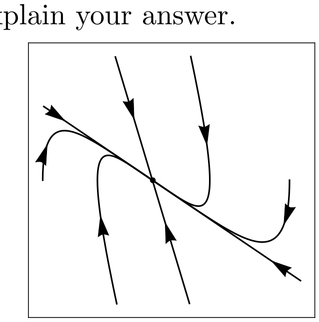
$$\frac{1}{9}\,r_1'' + \frac{1}{3}\,r_1' = k\,r_1 - \frac{1}{9}\,r_1' - \frac{1}{3}\,r_1 + f(t), \quad \text{or} \quad r_1'' + 4r_1' + (3 - 9k)r_1 = f(t).$$

b. [5 points] If $f(t) = \cos(\omega t)$ is the dashed curve in the figure below, for what values of k, if any, could the long-term behavior of r_1 be that shown by the solid curve? Explain your answer.



Solution: The solution for r_1 will be of the form $r_1 = r_c + r_p$, where $r_p = R\cos(\omega t - \delta)$. Thus, for this to be the long-term solution we must have $r_c \to 0$. This will be the case when the roots of the characteristic polynomial have negative real parts. The characteristic equation is $\lambda^2 + 4\lambda + (3 - 9k) = (\lambda + 2)^2 + (-1 - 9k) = 0$, for which $\lambda = -2 \pm \sqrt{1 + 9k}$. Thus we need $1 + 9k < 4$, or $k < 1/3$, for the real part of λ to be negative.

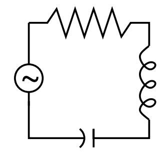
c. [5 points] If $f(t) = A_0$, a constant, for what values of k, if any, could the phase portrait for this system be similar to that shown in the figure below? Explain your answer.



Solution: Here we need the characteristic equation to have two negative, real roots. Thus we need $0 < 1 + 9k < 4$, so $-\frac{1}{9} < k < \frac{1}{3}$.

4. [13 points] Consider the RLC circuit shown to the right, below. This is modeled by $y'' + k\,y' + 2y = g(t)$, where $g(t)$ is the derivative of the input voltage and $0 < k < 2\sqrt{2}$ is proportional to the resistance of the resistor.

a. [9 points] If $g(t) = 4\cos(t)$, find the steady state response to the input. Write your answer in the form $R\cos(t - \alpha)$.



Solution: The steady state response will be the particular solution to the problem. Using the method of undetermined coefficients, let $y_p = A\cos t + B\sin t$. Then, plugging in and collecting terms in $\cos t$ and $\sin t$, we have

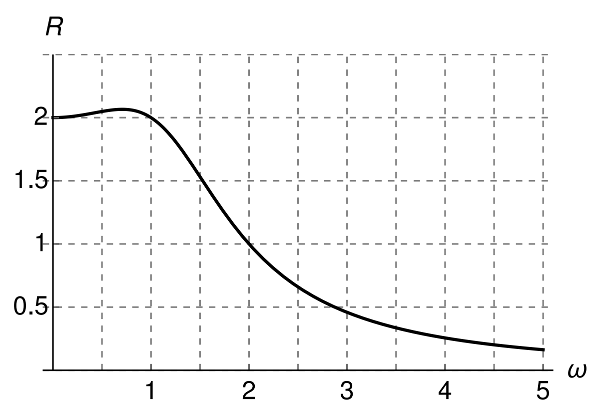
$$-A + Bk + 2A = 4, \quad \text{and} \quad -B - Ak + 2B = 0, .$$

These are $A + Bk = 4$, $-Ak + B = 0$. Solving by taking k times the first and adding, we have $(k^2+1)B = 4k$, so that $B = \frac{4k}{k^2+1}$. Then $A = \frac{4}{k^2+1}$. These are both positive, so in phase amplitude form we have

$$y_p = \sqrt{A^2 + B^2}\cos(t - \arctan(B/A)) = \frac{4}{\sqrt{k^2+1}}\cos(t - \arctan(k)).$$

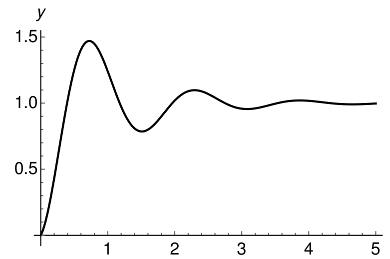
b. [4 points] The amplitude of the steady state response to the forcing $g(t) = 4\cos(\omega t)$ is shown below, as a function of ω. What is the value of k in the equation? Why?

Solution: The expression for R above is for $\omega = 1$, so $\frac{4}{\sqrt{k^2+1}} = 2$, where we have read the value for R from the figure. This gives $\sqrt{k^2+1} = 2$, so $k^2 + 1 = 4$ and $k = \pm\sqrt{3}$. Given that it is an RLC circuit we can discard the negative value, taking $k = \sqrt{3}$.



5. [14 points] For the first two of the following, identify each as true or false, by circling "True" or "False" as appropriate, and provide a short (one sentence) explanation indicating why you selected that answer. For the last give a short answer explaining the indicated question.

a. [4 points] For some constant ω and k, a solution to the mechanical system $y'' + 2y' + ky = \cos(\omega t)$ could be that shown to the right.



Solution: This cannot be true; the forcing requires that the steady state solution be sinusoidal, and centered on the t-axis.

True [False]

b. [4 points] Let $F(s) = \frac{s^2+1}{s^2+3s+5}$. There is some piecewise continuous function $f(t)$, of exponential order, for which $\mathcal{L}\{f(t)\} = F(s)$.

True [False]

Solution: This is false, because $F(s) \to 1 \neq 0$ as $s \to \infty$. We know that all transforms of regular functions must go to zero as $s \to \infty$.

c. [6 points] Your friends Anna and Andrew are solving the two problems $y'' + 0.1y' + y = 0$, $y(0) = 0$, $y'(0) = 1$ and $z'' + 0.1z' + z = \delta(t - 3)$, $z(0) = 0$, $z'(0) = 0$. Anna thinks that $z(t) = y(t - 3)$, while Andrew thinks they are different. Explain why they are both partly correct.

Solution: Note that the transforms of these problems give $Y = 1/(s^2 + 0.1s + 1)$ and $Z = e^{-3s}/(s^2 + 0.1s + 1)$. Thus we know that $z(t) = y(t - 3)u_3(t)$. The two are the same, with the ambiguity of the value of the derivative at $t = 3$—because z has the step function there the value of z' at $t = 3$ is not uniquely determined.

6. [14 points] Find solutions to each of the following, as indicated.

a. [7 points] Find the general solution to $y'' + y' = \dfrac{1}{1+e^t}$. *(Hint:* $\int \frac{1}{1+e^t}\,dt = t - \ln(1+e^t)$.*)*

Solution: Solutions to the homogeneous problem are $y_1 = 1$ and $y_2 = e^{-t}$. We cannot here use the method of undetermined coefficients, and so use variation of parameters instead. Let $y_p = u_1 + u_2 e^{-t}$. Then

$$u_1' + u_2' e^{-t} = 0 \quad \text{and} \quad -u_2' e^{-t} = \frac{1}{1+e^t}.$$

The second equation gives $u_2' = -e^t(1+e^t)^{-1}$, so that $u_2 = -\ln(1+e^t)$. The first then gives $u_1' = -u_2' e^{-t} = 1/(1+e^t)$. Following the hint, $u_1 = t - \ln(1+e^t)$. Thus the general solution is

$$y = c_1 + c_2 e^{-t} + (t - \ln(1+e^t)) - e^{-t}\ln(1+e^t).$$

b. [7 points] Find the solution to $y'' - y' = u_2(t) - u_3(t)$, $y(0) = 0$, $y'(0) = 0$

Solution: We use Laplace transforms: transforming both sides of the equation, we have

$$s^2 Y - sY = \frac{1}{s}(e^{-2s} - e^{-3s}), \quad \text{so} \quad Y = \frac{1}{s^2(s-1)}(e^{-2s} - e^{-3s}).$$

To find the inverse transform of this, we first apply partial fractions to the rational function. Letting $\frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$, we have after clearing the denominator

$$1 = As(s-1) + B(s-1) + Cs^2.$$

Setting $s = 0$, we have $B = -1$. If $s = 1$ we get $C = 1$. Finally, with $s = -1$, $1 = 2A + 2 + 1$, so that $A = -1$. Thus

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{s^2(s-1)}\} &= -\mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s^2}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\}\\ &= -1 - t + e^t = f(t).\end{aligned}$$

Thus, including the exponentials to get the expected step functions, we have

$$y = \mathcal{L}^{-1}\{Y\} = f(t-2)\,u_2(t) - f(t-3)\,u_3(t).$$

7. [14 points] Recall the linearized version of our laser model,

$$u' = -\gamma(Au + v)$$
$$v' = (A-1)\,u.$$

Consider the case of constant forcing (A = a constant) and the initial conditions $u(0) = u_0$, $v(0) = 0$.

a. [6 points] Find a system of algebraic equations for $U(s) = \mathcal{L}\{u(t)\}$ and $V(s) = \mathcal{L}\{v(t)\}$.

Solution: Transforming both sides of the system, we have

$$sU - u_0 = -\gamma A\,U - \gamma V$$
$$sV - 0 = (A-1)\,U,$$

or

$$(s + \gamma A)U + \gamma V = u_0$$
$$-(A-1)U + sV = 0.$$

b. [4 points] Solve your system from **(a)** to find $U(s)$ and $V(s)$.
(If you are unable to solve (a), consider the system $(s+a)U+bV = u_0$, $-cU+(s+a)V = 0$.)

Solution: Note that we have $V = \frac{A-1}{s}U$ from the second equation. Plugging this into the first, we have

$$(s+\gamma A)\,U + \frac{\gamma(A-1)}{s}\,U = \frac{s^2 + \gamma As + \gamma(A-1)}{s}U = u_0,$$

so that $U = \frac{s\,u_0}{s^2+\gamma As+\gamma(A-1)}$. Plugging this in for V, we have $V = \frac{(A-1)\,u_0}{s^2+\gamma As+\gamma(A-1)}$.
With the alternate system, we have $U = \frac{s+a}{c}V$, so that $((s+a)^2 + bc)V = u_0 c$, or $V = \frac{u_0 c}{(s+a)^2+bc}$, and $U = \frac{u_0(s+a)}{(s+a)^2+bc}$.

c. [4 points] Recall that in the lab we solved the characteristic equation $\lambda^2+\gamma A\lambda+\gamma(A-1) = 0$, finding $\lambda^2 + \gamma A\lambda + \gamma(A-1) = (\lambda + \frac{1}{2}\,\lambda A)^2 + (-\frac{1}{4}\gamma^2A^2 + \gamma(A-1)) = (\lambda+\mu)^2 + \nu^2 = 0$ (so that $\lambda = -\mu \pm i\nu$). Use this to rewrite your solutions in **(b)** in terms of μ and ν. Find $u(t) = \mathcal{L}^{-1}\{U(s)\}$ and $v(t) = \mathcal{L}^{-1}\{V(s)\}$ in terms of μ and ν.
(If you are stuck, assume the denominator of your U and V is of the form $(s+a)^2 + b^2$.)

Solution: We have

$$U = \frac{s\,u_0}{(s+\mu)^2+\nu^2} = \frac{(s+\mu)\,u_0}{(s+\mu)^2+\nu^2} - \frac{\mu\,u_0}{(s+\mu)^2+\nu^2}, \qquad V = \frac{(A-1)\,u_0}{(s+\mu)^2+\nu^2}.$$

Inverting these, we have

$$u(t) = u_0\,e^{-\mu t}\cos(\nu t) - \frac{u_0}{\nu}\,e^{-\mu t}\sin(\nu t), \qquad v(t) = \frac{(A-1)}{\nu}\,u_0\,e^{-\mu t}\sin(\nu t).$$