Math 216 — Final Exam

19 December, 2016

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [12 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Solve $2ty' + y = 5t^2$, $y(1) = 4$

Solution: This is the same as $y' + \frac{1}{2t}\,y = \frac{5}{2}\,t$, which is first order and linear. An integrating factor is $\mu = \exp(\int \frac{1}{2t}\,dt) = e^{\ln(t)/2} = t^{1/2}$. Multiplying through by μ and rewriting the left-hand side, we have $(\mu\, y)' = \frac{5}{2}\,t^{3/2}$. Integrating gives $t^{1/2}\,y = t^{5/2} + C$, so that $y = t^2 + C\,t^{-1/2}$. The initial condition requires that $C + 1 = 4$, so that $C = 3$, and

$$y = t^2 + 3\,t^{-1/2}.$$

b. [6 points] Find the general solution to $2y'' + y' + 2y = t\,e^{-t}$.

Solution: The general solution will be $y = y_c + y_p$, where y_c is the solution to the complementary homogeneous problem. We take $y_c = e^{\lambda t}$, so that $2\lambda^2 + \lambda + 2 = 0$, or $\lambda^2 + \frac{1}{2}\lambda + 1 = 0$. Completing the square, $(\lambda + \frac{1}{4})^2 - \frac{1}{16} + 2 = 0$, and $\lambda = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}$. Thus $y_c = c_1 e^{-t/4}\cos(\sqrt{15}\,t/4) + c_2 e^{-t/4}\sin(\sqrt{15}\,t/4)$. To find y_p, we use undetermined coefficients and guess $y_p = (A + Bt)e^{-t}$. Then $y_p' = (-A + B)e^{-t} - Bt\,e^{-t}$, and $y_p'' = (A - 2B)e^{-t} + Bt\,e^{-t}$. Plugging into the original equation and collecting terms in e^{-t} and $t\,e^{-t}$, we have

$$\begin{aligned}(2(A - 2B) + (-A + B) + 2A)\,e^{-t} + (2B - B + 2B)\,t\,e^{-t} &= t\,e^{-t}, \quad \text{or}\\ (3A - 3B)e^{-t} + 3B\,t\,e^{-t} &= t\,e^{-t}.\end{aligned}$$

Thus $B = \frac{1}{3}$, and $A = \frac{1}{3}$. Our general solution is

$$y = c_1 e^{-t/4}\cos(\sqrt{15}\,t/4) + c_2 e^{-t/4}\sin(\sqrt{15}\,t/4) + \left(\frac{1}{3} + \frac{1}{3}\,t\right)e^{-t}.$$

2. [12 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the general solution to the system $x' = x + 2y$, $y' = 6x + 2y$.

Solution: In matrix form, this is $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 6 & 2 \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are determiend by

$$(1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0,$$

so $\lambda = -2$ and $\lambda = 5$. If $\lambda = -2$ the eigenvector satisfies $3v_1 + 2v_2 = 0$, so $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, and if $\lambda = 5$, we have $-4v_1 + 2v_2 = 0$, so $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}.$$

b. [6 points] Solve $y'' + 4y' + 4y = u_2(t)e^{-3(t-2)}$, $y(0) = 0$, $y'(0) = 7$.

Solution: Because of the step forcing, we opt for Laplace transforms. Transforming both sides of the equation, we have, with $Y = \mathcal{L}\{y\}$,

$$s^2Y - 7 + 4sY + 4Y = \frac{e^{-2s}}{s+3},$$

so that $Y = 7/(s+2)^2 + e^{-2s}/((s+3)(s+2)^2)$. Note that the first of these inverts easily as $\mathcal{L}^{-1}\{7/(s+2)^2\} = 7t\,e^{-2t}$. To invert the second, use partial fractions on the non-exponential part:

$$\frac{1}{(s+3)(s+2)^2} = \frac{A}{s+3} + \frac{B}{s+2} + \frac{C}{(s+2)^2},$$

so that, clearing the denominators,

$$1 = A(s+2)^2 + B(s+3)(s+2) + C(s+3).$$

With $s = -2$ we find $C = 1$, and with $s = -3$, $A = 1$ also. Then if $s = 0$, $1 = 4 + 6B + 3$, and $B = -1$. Thus, inverting and applying the exponential rule, we have

$$\mathcal{L}^{-1}\{\frac{e^{-2s}}{(s+3)(s+2)^2}\} = u_2(t)\,(e^{-3(t-2)} - e^{-2(t-2)} + (t-2)\,e^{-2(t-2)}).$$

Our complete solution is therefore

$$y = 7t\,e^{-2t} + u_2(t)\,(e^{-3(t-2)} - e^{-2(t-2)} + (t-2)\,e^{-2(t-2)}).$$

3. [16 points] Consider a model for interacting populations x_1 and x_2 given by

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

a. [2 points] What type of interaction do you think there is between these populations (how does the interaction affect each population)? Explain.

Solution: We expect that x_1 is a prey population and x_2 a predator. The interaction terms are $-\frac{x_1x_2}{3+x_1}$ for x_1 and $\frac{x_1x_2}{3+x_1}$ for x_2. Thus x_1 is disadvantaged by the interaction, while x_2 is advantaged. Further, in the absence of x_1, the population of x_2 will exponentially decay to zero.

b. [4 points] Find all critical points for this system.

Solution: Setting derivatives to zero, the second equation gives $x_2(-1 + \frac{2x_1}{3+x_1}) = 0$, so $x_2 = 0$ or $2x_1 = 3 + x_1$, so $x_1 = 3$. If $x_2 = 0$, the first equation requires $x_1 = 0$. If $x_1 = 3$, the first equation becomes $0 = 6 - 2x_2$, so that $x_2 = 3$. Thus there are two critical points, $(0, 0)$ and $(3, 3)$.

c. [6 points] The Jacobian for this system is $J(x_1, x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each. *(This problem part continues on the next page.)*

Problem 3, continued. We are considering the system

$$x_1' = 2x_1 - \frac{4x_1x_2}{3+x_1}, \quad x_2' = -x_2 + \frac{2x_1x_2}{3+x_1}.$$

c. *Continued: we are solving the problem*

The Jacobian for this system is $J(x_1,x_2) = \begin{pmatrix} 2 - \frac{12x_2}{(x_1+3)^2} & -\frac{4x_1}{x_1+3} \\ \frac{6x_2}{(x_1+3)^2} & -1 + \frac{2x_1}{x_1+3} \end{pmatrix}$. Classify each of your critical points from (b) by stability and type, and sketch a phase portrait for each.

Solution: From the given Jacobian, we have at at the critical point $(0,0)$, $J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, so that $\lambda = 2, -1$, and we see that the origin is an unstable saddle point.

At $(3,3)$, we have $J(3,3) = \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & 0 \end{pmatrix}$, so that eigenvalues satisfy $\lambda^2 - \lambda + 1 = (\lambda - \frac{1}{2})^2 + \frac{3}{4} = 0$, and $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\,i$. Thus this is unstable spiral source.

(Phase portraits are omitted here, but are as expected.)

d. [4 points] If the population of x_1 was initially large and that of x_2 small, sketch a qualitatively accurate graph of x_1 and x_2 as functions of time. What happens to the populations for large times?

Solution: Because of the context, we would initially guess that the population of x_1 will decrease and that of x_2 will increase, and that the populations will oscillate with decreasing magnitude as they converge to $x_1 = x_2 = 3$. However, this isn't what our analysis above suggests: because the critical point $(3,3)$ is an unstable spiral, we expect that the trajectory will either move around $(3,3)$ (possibly more than once) before heading out along the x_1 axis and increasing to (∞,∞), or will fail to oscillate at all and move to the same limiting behavior.

(The solution curves are graphs of x_1 and x_2 against time.)

4. [10 points] Consider the system

$$x' = -10x + 10y, \quad y' = r\,x - y - xz, \quad z' = 4z + xy.$$

For $r > 1$ this has a critical point $P = (2\sqrt{r-1}, 2\sqrt{r-1}, r-1)$. Let $\mathbf{A}$ be the matrix that gives the linearization of this system at P, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

a. [6 points] When $r = 34$, eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = -15$, $\lambda_2 = -13.3i$, and $\lambda_3 = 13.3i$, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0.4 - 0.3i \\ -0.8i \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0.4 + 0.3i \\ 0.8i \\ 1 \end{pmatrix}$.

Write a real-valued general solution to the linearization of the system in this case.

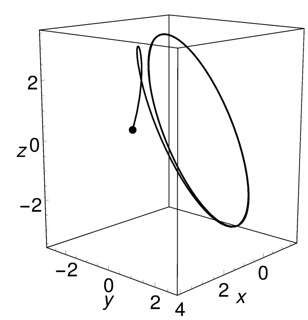
Solution: To write the general solution we need three linearly independent solutions; one is $\mathbf{v}_1\, e^{\lambda_1 t}$. To find two other (real-valued) solutions, we separate the real and imaginary parts of $\mathbf{v}_3\, e^{\lambda_3 t}$ (for brevity, we write $\mu = 13.3$):

$$\begin{aligned} \mathbf{v}_3\, e^{\lambda_3 t} &= \begin{pmatrix} (0.4 + 0.3i) \\ 0.8i \\ 1 \end{pmatrix} (\cos(\mu t) + i\sin(\mu t)) \\ &= \begin{pmatrix} (0.4\cos(\mu t) - 0.3\sin(\mu t)) + i(0.3\cos(\mu t) + 0.4\sin(\mu t)) \\ -0.8\sin(\mu t) + 0.8i\cos(\mu t) \\ \cos(\mu t) + i\sin(\mu t) \end{pmatrix}. \end{aligned}$$

Thus a general solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} e^{-15t} + c_2 \begin{pmatrix} 0.4\cos(\mu t) - 0.3\sin(\mu t) \\ -0.8\sin(\mu t) \\ \cos(\mu t) \end{pmatrix} + c_3 \begin{pmatrix} 0.3\cos(\mu t) + 0.4\sin(\mu t) \\ 0.8\cos(\mu t) \\ \sin(\mu t) \end{pmatrix}.$$

b. [4 points] Could the phase space trajectory shown to the right be that for the linearized system from (a)? Could it be that for the nonlinear system? Explain.



Solution: This certainly could correspond to the linearized system: from our solution in (a), we expect the component of the solution in the direction of $\mathbf{v}_1$ to collapse to zero, leaving a purely oscillatory planar solution; this is shown here. For the nonlinear system it is less clear, as we do not know for certain that the behavior of the linearized system will correctly approximate the nonlinear system when we get purely imaginary eigenvalues.

5. [12 points] Consider the nonlinear system

$$x' = y, \quad y' = -3x - 2y + r\,x^2.$$

Four possible phase portraits for this system are shown along the right side of the page.

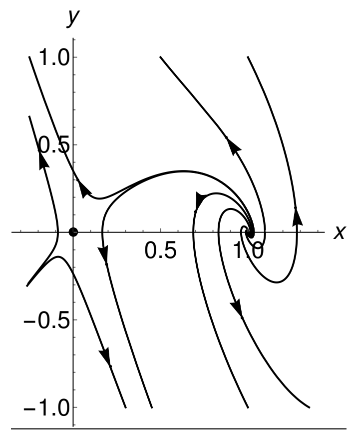
a. [4 points] If one of the graphs is to match this system, what is the value of the parameter r? Why?

Solution: We note that critical points for the system are given by $y = 0$, $-3x - 2y + r\,x^2 = 0$, or $x(-3 + r\,x) = 0$. Thus the critical points are $(0, 0)$ and $(3/r, 0)$. From the phase portraits we know that the critical points are $(0, 0)$ and $(1, 0)$, so $r = 3$.

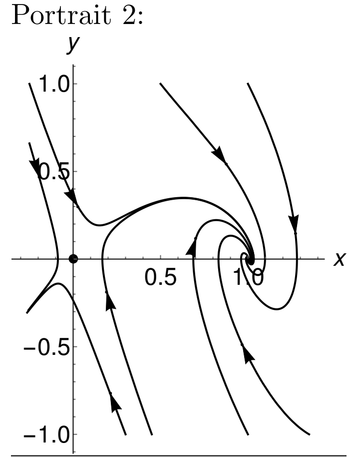
b. [8 points] Given the value of r you found in **(a)**, which, if any, of the phase portraits could be that for this system? Why?

Solution: This is most easily determined by finding the linear behavior near the critical points. The Jacobian for the system is $J = \begin{pmatrix} 0 & 1 \\ -3 + 6x & -2 \end{pmatrix}$, so at $(0, 0)$ we have $J(0, 0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$, and at $(1, 0)$, $J(1, 0) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$. The eigenvalues of the corresponding systems are given by $\lambda^2 + 2\lambda \pm 3 = 0$. For $(0, 0)$, $\lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0$, so $\lambda = -1 \pm i\sqrt{2}$, and the critical point is a spiral sink. We note also that to the right of $(0, 0)$, $u' = 0$ and $v' < 0$, so rotation is clockwise.
For $(1, 0)$, $\lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0$, so that $\lambda = -3$ and $\lambda = 1$, and the point is a saddle point. We note also that when $\lambda = -3$, $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and when $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Thus phase portrait 3 could be correct.

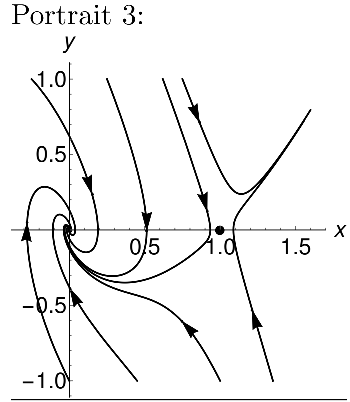
Portrait 1:



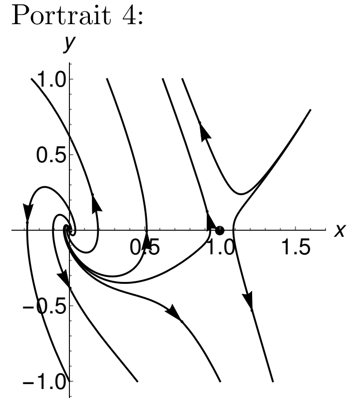
Portrait 2:



Portrait 3:



Portrait 4:



6. [12 points] For the following, identify each as true or false by circling "True" or "False" as appropriate. Then, if it is true, provide a short (one sentence) explanation indicating why it is true; if false, explain why or provide a counter-example.

a. [3 points] Let $\mathbf{A}$ be a 3×3 matrix with characteristic polynomial $p(\lambda) = \lambda^3 + 4\lambda^2 + \lambda - 6$. Then the origin is an asymptotically stable critical point of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

True $\boxed{\text{False}}$

Solution: We note that $p(1) = 1 + 4 + 1 - 6 = 0$, so $\lambda = 1$ is an eigenvalue. Thus there is a solution $\mathbf{x} = \mathbf{v}e^t$ to the system, and it is clearly unstable.

b. [3 points] Consider the equation $y' = f(t, y)$, with f continuous for all values of t and y. We can solve this either by using an integrating factor or by separating variables (though in the latter case we may not be able to get an explicit solution for y).

True $\boxed{\text{False}}$

Solution: This is false. Consider $y' = y^2 + x$. This is neither linear (so integrating factors do not work) nor separable.

c. [3 points] While we cannot solve the nonlinear system $x' = x - x^2 - xy + \sin(t)$, $y' = y + xy$, we can obtain a good qualitative understanding of solutions by linearizing around critical points and sketching a phase portrait.

True $\boxed{\text{False}}$

Solution: Because this is not autonomous we will be unable to find equilibrium solutions (that is, critical points), and the phase portrait will depend on time.

d. [3 points] Long-term solutions to the system $y'' + 4y = 3\cos(4t)$ will be periodic.

$\boxed{\text{True}}$ False

Solution: The solution to this will be $y_c = C\cos(2t - \delta_1) + R\cos(4t - \delta_2)$, which will be periodic with period π.

7. [12 points] Suppose that we are considering a system of two linear, constant-coefficient differential equations for x_1 and x_2 given in matrix form by $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}$. We know that eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$, $\lambda_2 = -3$.

a. [4 points] Suppose that $\mathbf{g} = \mathbf{0}$. If we rewrite the system as a single second-order linear equation in one of x_1 or x_2, what is the equation?

Solution: Because we know that the characteristic polynomial for the system and the second-order equation must be same, we know that it must be $p(\lambda) = (\lambda - 1)(\lambda + 3)) = \lambda^2 + 2\lambda - 3$, so the equation must be

$$y'' + 2y' - 3y = 0.$$

b. [4 points] Suppose now that $\mathbf{g}$ is nonzero, and let $x_1(0) = x_2(0) = 0$. If we apply the Laplace transform to the system and solve for $X_1 = \mathcal{L}\{x_1\}$ we will get $X_1 = G(s)/H(s)$, where $G(s)$ is a transform involving the components of $\mathbf{g}$. What is $H(s)$? Explain how you know.

Solution: From above, we know that the characteristic polynomial of the system is $s^2 + 2s - 3$; thus, we know that

$$X_1(s) = \frac{G(s)}{(s^2 + 2s - 3)},$$

where $G(s)$ is some combination of the transforms of the components of $\mathbf{g}$. Thus $H(s) = s^2 + 2s - 3$. This makes sense given the transfer function of the corresponding second order equation must be one over the characteristic polynomial evaluated on s.

c. [4 points] Finally suppose the eigenvectors of $\mathbf{A}$ are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and we solve the original system with the forcing term $\mathbf{g} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What is $\mathbf{x}_c$? What will the general solution look like? Specify all functions in your answer. (You may leave your solution in terms of $\mathbf{A}$ if you provide the matrix equation you would need to determine it completely.)

Solution: If the forcing term is constant, we would guess that the particular solution is a constant solution $\mathbf{c}$, which will then satisfy $\mathbf{A}\mathbf{c} = -\mathbf{g}$. The general solution will then be

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \mathbf{c}.$$

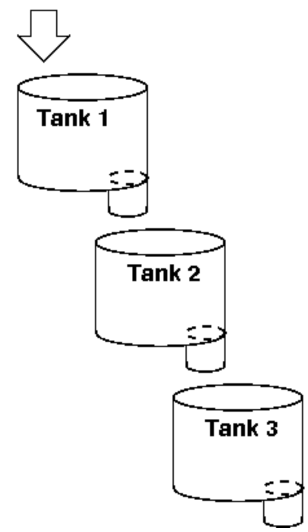
8. [7 points] Consider a three tank system as suggested by the figure to the right, below. The volumes of the three tanks are V_1, V_2 and V_3; suppose that they are initially full of water, and that there is x_0 kg of a contaminant in the first tank. Pure water is added to tank 1 at a rate of r liters/second, and the well-mixed mixture moves at the same rate from tank 1 to tank 2, from tank 2 to tank 3, and out of tank 3. Let x_1, x_2 and x_3 be the amount of the contaminant in tanks 1, 2, and 3. Write a system of differential equations, in matrix form, for x_1, x_2 and x_3. Indicate the initial condition that completes the initial value problem.

Solution: For each tank we have $x_j' = (\text{rate in})-(\text{rate out})$. For tank 1, rate in is zero, and the rate out is (concentration)(rate) $= \frac{x_1}{V_1}\,r = \frac{r}{V_1}\,x_1$. Thus for tank 1 we have $x_1' = -\frac{r}{V_1}\,x_1$.
For tank 2, the rate in is the output from tank 1, and using similar logic as there we have $x_2' = \frac{r}{V_1}\,x_1 - \frac{r}{V_2}\,x_2$. Similarly, $x_3' = \frac{r}{V_2}\,x_2 - \frac{r}{V_3}\,x_3$. Thus, in matrix form, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -\frac{r}{V_1} & 0 & 0 \\ \frac{r}{V_1} & -\frac{r}{V_2} & 0 \\ 0 & \frac{r}{V_2} & \frac{r}{V_3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

with

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix}.$$



9. [7 points] Consider the mechanical system $y'' + ky' + 3y = g(t)$, $y(0) = 1$, $y'(0) = 3$. Find a $g(t)$ and all values of k for which of the following will be true:
(a) the steady state response of the system will be purely sinusoidal with period π;
(b) the response to the initial conditions will have halved in amplitude by the time $t = 5$; and
(c) the system is underdamped.

Solution: For a steady state which is a sinusoid with period π we must have $k > 0$ and $g(t) = a\,\cos(2t) + b\,\sin(2t)$, for some a and b (not both zero).
Then the complementary homogeneous solution, which gives the response to the initial conditions, will have $\lambda^2 + k\lambda + 3 = 0$, so that $\lambda = -\frac{1}{2}\,k \pm \frac{1}{2}\sqrt{k^2 - 12}$. This is underdamped when $k^2 < 12$, in which case the homogeneous solution decays like $e^{-kt/2}$. Thus we need $e^{-5k/2} \le \frac{1}{2}$, or $k \ge -\frac{2}{5}\,\ln(1/2) = \frac{2}{5}\,\ln 2$. Combining this with the previous conditions for underdamping, we have

$$g(t) = a\,\cos(2t) + b\,\sin(2t), \qquad \frac{2}{5}\,\ln(2) \le k < \sqrt{12}.$$