Math 216 — First Midterm

19 October, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] The general solution to $x' = 3y$, $y' = 2x + y$

b. [8 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & -4 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

2. [16 points] Let $\mathbf{A}$ be a 2×2 matrix with real entries that has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a. [6 points] What is the result of each of the following matrix multiplications? Briefly explain your answer for each.

$\mathbf{A} \begin{pmatrix} -1 \\ 1 \end{pmatrix} =$

$\mathbf{A} \begin{pmatrix} 2 \\ 0 \end{pmatrix} =$

b. [5 points] Sketch a qualitatively accurate phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

c. [5 points] Give two initial conditions for which the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will, as trajectories in the phase plane, eventually be parallel to the line $y = -x$. Give a short explanation of how you know your answer is correct.

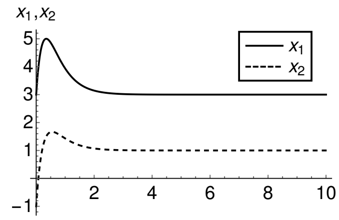
3. [14 points] Consider the systems of equations below. In each the vector $\mathbf{x}$ has components x_1 and x_2.

A. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x}$

B. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Match each of these to one of the graphs to the right (note that two of these are component plots and two are phase portraits). Briefly explain how you know that your matching is correct.

I.



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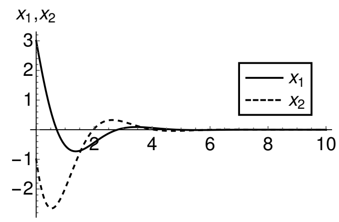

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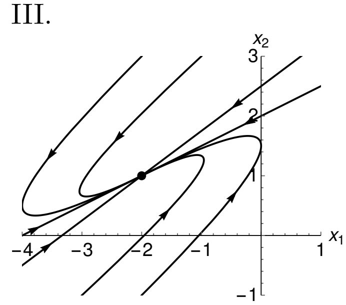

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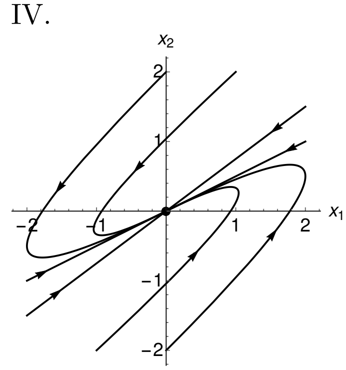


4. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] Find the general solution to $y' = 7 - \dfrac{\cos(t)}{2+\sin(t)}y$.

b. [8 points] Solve the initial value problem: $\dfrac{y}{t^2+5}\,y' = 1,\ y(0) = -2$

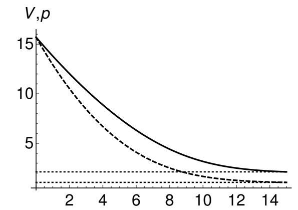
5. [14 points] The volumetric rate at which liquid leaves a cylindrical tank through a circular hole in its bottom is proportional to the square root of the volume of liquid in the tank. Suppose we have a cylindrical tank 5 meters tall with a 1 meter radius (so that its volume is 5π m^3) that is initially full of some liquid. At time $t = 0$, a circular hole opens in the base and more liquid is added at a rate of 1 m^2/hr.

a. [5 points] Write an initial value problem for the volume V of liquid in the tank. (Your answer will involve a constant of proportionality k.) Can you solve your equation? Explain. *(Do not actually solve the equation.)*

b. [5 points] Suppose that the solution to your equation in (a) is some function $V(t)$. If the liquid in the tank initially contains a particulate at a concentration of 1 g/m^3 and the liquid entering has a particulate concentration of 2 g/m^3, write an initial value problem for the amount of particulate in the tank. (Your answer will involve the unknown function $V(t)$.) Can you solve this equation? Explain. *(Do not actually solve the equation.)*

Problem 5, continued.

c. [4 points] What do you expect the long-term value for the volume $V(t)$ to be? Can you predict the long-term value for $p(t)$? If $k = 1$, which of the graphed functions to the right is $V(t)$ and which is $p(t)$? Why?



6. [10 points] Consider the initial value problem $(1 - y^3)\frac{dy}{dt} = 1$, $y(0) = 0$.

a. [5 points] Without solving it, will this initial value problem have a unique solution?

b. [5 points] Solve the problem. Based on your solution, for what range of t and y values would you expect the solution to exist? Why?

7. [16 points] In lab we considered the van der Pol system $x' = y$, $y' = -x - \mu y \frac{df}{dx}$. Here, we suppose that $f'(x) = |x| - a$, so that this becomes $x' = y$, $y' = -x - \mu y\,(|x| - a)$.

a. [3 points] Find the critical point for this system.

b. [3 points] Linearize the system at your critical point from (a).

c. [5 points] Suppose that your linear system from (b) is, for some k that depends on both of μ and a, $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \mathbf{x}$. Determine the type and stability of the critical point.

Problem 7, continued. We are considering the system $x' = y$, $y' = -x - \mu y\,(|x| - a)$

d. [5 points] Each of the direction fields below is generated for the nonlinear equation we are considering, picking μ and a so that the value of k given in (c) is one of -2, -1, 1, or 2. Identify, with a short explanation, which graph corresponds to which value of k.

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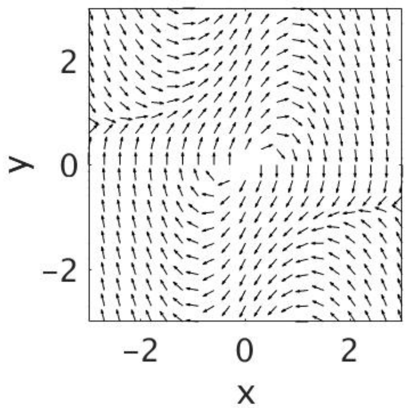

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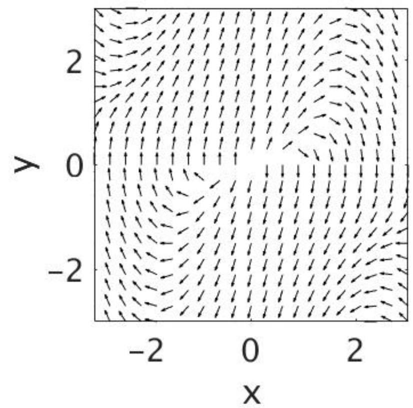

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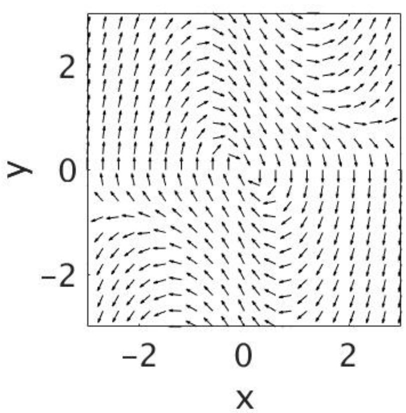

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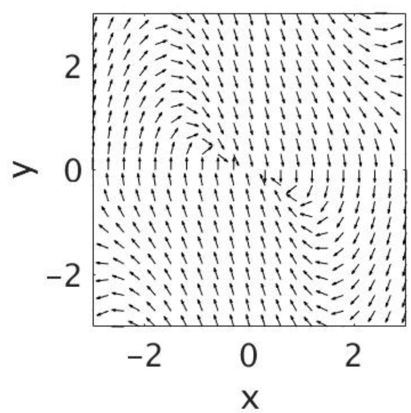

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