Math 216 — Second Midterm

16 November, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

---

1. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Find the solution to the initial value problem $y'' + 2y' + 17y = 0$, $y(0) = 1$, $y'(0) = 0$.

b. [8 points] Find the general solution to the equation $y'' + 5y' + 4y = e^t + 8t$.

2. [12 points] The following problems consider a non-homogeneous second-order linear differential equation $L[y] = g(t)$. Suppose that y_1 and y_2 are solutions to this equation, and that y_3, y_4, and y_5 are solutions to the complementary homogeneous problem $L[y] = 0$.

a. [3 points] Can you say what problem each of the following solve? If so, indicate what it is; if not, write "none." (No explanation necessary.)

i. $y_1 - y_2$

ii. $y_1 - y_3$

iii. $y_1 + y_2 + y_3 + y_4 + y_5$

b. [3 points] Explain how you are able to determine your answers in (a), or why it is not possible to tell.

c. [6 points] The following statements are not guaranteed to be true. Explain why.

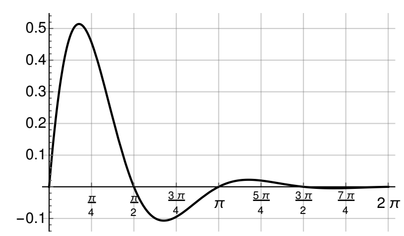
i. The solution to the initial problem $L[y] = 0$, $y(0) = y_0$, $y'(0) = v_0$ (for any y_0 and v_0) can be written as $y = c_1y_3 + c_2y_4 + c_3y_5$ for some c_1, c_2, and/or c_3 (where one or more of c_1, c_2, and c_3 may be zero).

ii. Because both y_1 and y_2 satisfy $L[y] = g(t)$, we must have $y_1 = y_2$.

3. [15 points] For all of the following, the equations are linear, constant-coefficient, and second-order, with the coefficient of y'' picked to be one.

a. [5 points] If the differential equation is nonhomogeneous and the general solution is $y = c_1e^{-2t} + c_2e^{-3t} + 4\cos(2t)$, what is the differential equation?

b. [5 points] If the graph to the right shows the movement of a unit mass on a spring with damping constant 2, set in motion with an initial velocity of 1 m/s, write an initial value problem modeling the position of the mass.



c. [5 points] Consider the (linear, constant-coefficient. . .) equation $L[y] = 0$ and the equivalent system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If one solution to the equation $L[y] = 0$ is $y = e^{-t}$, what is a corresponding solution to the system? If the coefficient of y in the equation is 3, what is the differential equation?

4. [14 points] Recall that the nonlinear model for the number of photons P and population inversion N in a ruby laser that we considered in lab 3 had an equilibrium point $(P, N) = (1, A-1)$. If we assume that $A = A_0 + a\cos(\omega t)$, with a a very small value, the dynamics of the system near the equilibrium point are modeled by the linear system $u' = -\gamma(Au + v) + \gamma a\cos(\omega t)$, $v' = (A-1)u$

a. [4 points] Rewrite this linear system as a second order equation in v.

b. [6 points] Suppose that the second order equation you obtain in (a) is $v'' + 2v' + v = \cos(\omega t)$, so that the solution to the complementary homogeneous problem is $v_c = c_1e^{-t} + c_2te^{-t}$. Set up the solution for v_p using variation of parameters, and solve them to obtain explicit equations for u_1' and u_2' in terms of t only. *(Do not solve these to find u_1 and u_2.)*

c. [4 points] It turns out that, for some A and B, $v_p = A\cos(\omega t) + B\sin(\omega t)$. Representative values of A and B are given for different values of ω in the table below. Does the system exhibit resonance? Write the response v_p to a forcing of $\cos(2t)$ in phase-amplitude form.

$\omega =$	1	2	3	4	5
$A =$	0	-0.12	-0.08	-0.06	-0.04
$B =$	1	0.64	0.36	0.22	0.15

5. [14 points] Consider the initial value problem $y'' + 4y' + 4y = 9e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

a. [6 points] Solve the problem *without* using Laplace transforms.

b. [8 points] Solve the problem *with* Laplace transforms.

6. [15 points] Complete each of the following problems having to do with the Laplace transform.

a. [5 points] Find the inverse Laplace transform of $F(s) = \dfrac{5s}{s^2 + 4s + 6}$

b. [5 points] Given that $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to derive the transform rule $-F'(s) = \mathcal{L}\{tf(t)\}$.

c. [5 points] Consider the initial value problem $ty'' + y = 0$, $y(0) = 1$, $y'(0) = 0$. If $Y = \mathcal{L}\{y\}$, what equation does Y satisfy?

7. [15 points] Consider the system of differential equations $x' = 3x + 4y$, $y' = 2x + y$, with initial conditions $x(0) = 0$, $y(0) = 2$.

a. [6 points] Using Laplace transforms, find explicit equations for $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

b. [4 points] Find x and y in terms of any constants you may have in partial fractions expansions of X and Y (that is, do not solve for the values of those constants).

c. [5 points] If we rewrote the system as a second order differential equation $L[y] = 0$ for y, what would the characteristic equation for λ be? What is the linear operator L?

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$

This page provided for additional work.

This page provided for additional work.