Math 216 — Final Exam

14 December, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Solve the initial value problem $2ty' - y = -3t^2$, $y(1) = 2$. (Consider $t \geq 1$.)

b. [8 points] Find the general solution to $y'' - 4y = 12t + e^{-2t}$.

2. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Find the general solution to the system $x' = 2x + 3y$, $y' = 5x + 4y$.

b. [8 points] Solve the initial value problem $y'' - 4y' + 13y = -\delta(t-4)$, $y(0) = 1$, $y'(0) = 3$.

3. [12 points] For parts (a) and (b), identify each as true or false, and give a short mathematical calculation, with explanation, justifying your answer. For part (c) the statement is false. Explain why.

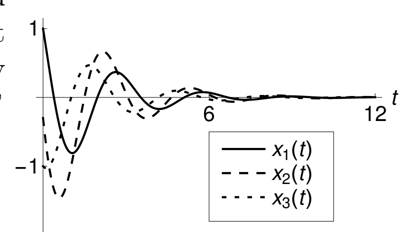
a. [4 points] If L is a linear second-order differential operator, y_1 and y_2 are non-zero functions for which $L[y_1] = L[y_2] = 0$, and y_3 is a function for which $L[y_3] = \frac{3}{2+t^2}$, then for any c_1, c_2, and c_3, $y = c_1 y_1 + c_2 y_2 + c_3 y_3$ solves $L[y] = \frac{3}{2+t^2}$.

True False

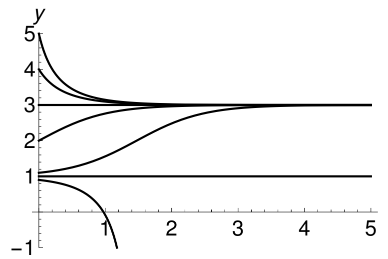
b. [4 points] If L is a linear second-order differential operator with continuous coefficients, and y_1 and y_2 are non-zero functions satisfying $L[y] = 0$, $y_1(0) = y_2'(0) = 0$ and $y_1'(0) = y_2(0) = 1$, then a general solution to $L[y] = 0$ is given by $y = c_1 y_1 + c_2 y_2$.

True False

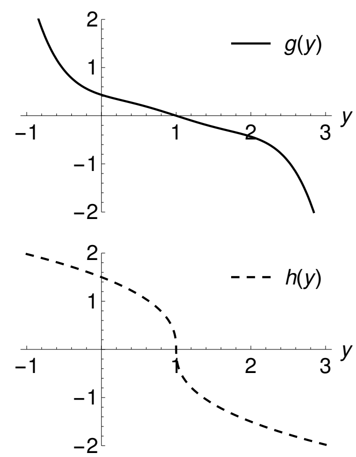
c. [4 points] Suppose $\mathbf{A}$ is a real-valued 3×3 matrix, and that the three curves shown to the right are the component plots of a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, as indicated. Explain why the statement "eigenvalues of $\mathbf{A}$ must be complex-valued" is false.



4. [6 points] Representative solution curves for a first-order differential equation $y' = f(y)$ are shown in the figure to the right. Write a possible function $f(y)$ that could give this behavior. Explain why your function could be correct.



5. [6 points] Consider the two equations
A. $y' = g(y)$ and
B. $y' = h(y)$,
where $g(y)$ and $h(y)$ are given in the figures to the right. Which of the following three statements must hold for each equation?
1. "The equation with the initial condition $y(0) = 1$ must have a unique solution."
2. "The equation with an initial condition $y(0) = y_0 < 1$ must have a unique solution."
3. "The solution to the equation with an initial condition $y(0) = y_0 < 1$ will asymptotically approach $y = 1$ as $t \to \infty$."
Briefly explain your answers.



6. [11 points] For parts (a) and (b) explain what is wrong with the calculation. For part (c), show the indicated result.

a. [4 points] *(Explain why this is incorrect:)* $\mathcal{L}^{-1}\{\frac{4e^{-3s}}{(s^2+4)}\} = 2u_3(t)\,\sin(2t)$

b. [4 points] *(Explain why this is incorrect:)* If $f(t) = \begin{cases} 2t, & t \leq 3 \\ 5, & t > 3 \end{cases}$, then $\mathcal{L}\{f(t)\} = \frac{2}{s^2} + \frac{5e^{-3s}}{s}$.

c. [3 points] Use convolution and properties of $\delta(t-c)$ to show that $\mathcal{L}^{-1}\{e^{-sc}F(s)\} = u_c(t)f(t-c)$, given that $\mathcal{L}\{f(t)\} = F(s)$. *(Note: this may be challenging, and you may want to leave it until you've worked other problems on the final.)*

7. [9 points] Consider the system $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The coefficient matrix has eigenvalues and eigenvectors $\lambda = -2$ and $\lambda = \pm i$, with $\mathbf{v}_{-2} = \begin{pmatrix} -5 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{v}_{\pm i} = \begin{pmatrix} 0 \\ 1 \\ \pm i \end{pmatrix}$.

a. [6 points] Write a general, real-valued solution to the system.

b. [3 points] As $t \to \infty$, what happens to solution trajectories? If we look at trajectories for large times, what will we see?

8. [12 points] Suppose that for some nonlinear second-order differential equation $y'' = f(y)$ we can write an equivalent system of two first-order differential equations $x_1' = F(x_1, x_2)$, $x_2' = G(x_1, x_2)$. Critical points of the latter are $\mathbf{x}_0 = (0, 0)$ and $\mathbf{x}_1 = (1, 0)$. The Jacobian at these points is $\mathbf{J}(\mathbf{x}_0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{J}(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$.

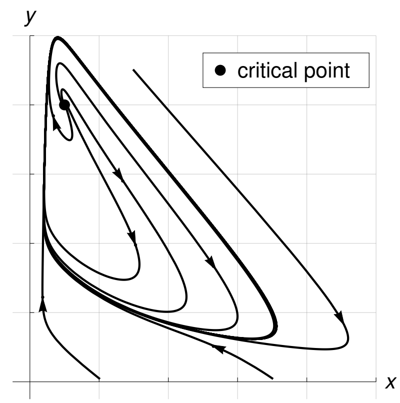
a. [8 points] Sketch a phase portrait for the nonlinear system.

b. [4 points] Based on your phase portrait, sketch a qualitatively accurate graph of y as a function of t if we start with the initial condition $y(0) = 0$, $y'(0) = 1$.

9. [14 points] The Brusselator is a nonlinear model of a chemical reaction which can have oscillatory concentrations x and y of the chemicals in the reaction. A model for this is

$$x' = 1 - (b+1)x + \frac{1}{4}x^2y, \quad y' = bx - \frac{1}{4}x^2y.$$

The figure to the right gives the phase portrait for this system for some value of b.



a. [4 points] What are the coordinates of the critical point shown? *(Note that your answer may involve the parameter b.)*

b. [7 points] Given the behavior shown in the phase portrait, what can you say about the parameter b?

c. [3 points] We said that the Brusselator can have oscillatory concentrations of x and y. Explain how the result here does (or does not) demonstrate this behavior.

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Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$
F.	$\int_0^t f(x)g(t-x)\,dx$	$F(s)G(s)$