Math 216 — First Midterm

19 October, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] The general solution to $x' = 3y$, $y' = 2x + y$

Solution: As a matrix equation, this is $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$. Eigenvalues of $\mathbf{A}$ are given by

$$\det(\begin{pmatrix} -\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}) = -\lambda(1-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.$$

Thus $\lambda = -2, 3$. If $\lambda = -2$, the eigenvector satisfies $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Similarly, if $\lambda = 3$, we have $\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

b. [8 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ 2 & -4 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Solution: Eigenvalues of the coefficient matrix satisfy

$$\det(\begin{pmatrix} -\lambda & -2 \\ 2 & -4-\lambda \end{pmatrix}) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0,$$

so $\lambda = -2$, twice. Eigenvectors satisfy

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0},$$

so $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To find a second solution, we look for $\mathbf{x} = (t\mathbf{v} + \mathbf{w})e^{-2t}$, so that

$$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{w} = \mathbf{v}.$$

This is $2w_1 - 2w_2 = 1$, so we may take $w_1 = \frac{1}{2}$ and $w_2 = 0$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) e^{-2t}.$$

Plugging in the initial condition, we have $c_1 + c_2/2 = 3$, and $c_1 = 0$. Thus $c_2 = 6$, and

$$\mathbf{x} = 6 \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right) e^{-2t}.$$

2. [16 points] Let $\mathbf{A}$ be a 2×2 matrix with real entries that has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a. [6 points] What is the result of each of the following matrix multiplications? Briefly explain your answer for each.

$\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix} =$

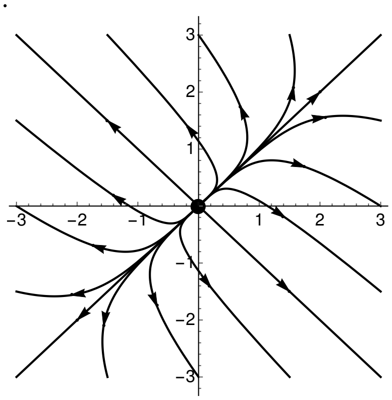
$\mathbf{A}\begin{pmatrix} 2 \\ 0 \end{pmatrix} =$

Solution: Note that $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector—in fact, it is $-\mathbf{v}_2$. Thus $\mathbf{A}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$.

Then note that $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus $\mathbf{A}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \mathbf{A}\mathbf{v}_1 + \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 5\mathbf{v}_2 = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$.

b. [5 points] Sketch a qualitatively accurate phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Solution: The equilibrium solution is $(0, 0)$, and there are two straight-line solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$, both with outward trajectories. The eigenvalue associated with $\mathbf{v}_2$ is much larger than the other, so trajectories that are not on $k\mathbf{v}_1$ will bend to be parallel to $\mathbf{v}_2$, as shown in the figure below.



c. [5 points] Give two initial conditions for which the solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will, as trajectories in the phase plane, eventually be parallel to the line $y = -x$. Give a short explanation of how you know your answer is correct.

Solution: Note that the line $y = -x$ is the line given by the second eigenvector. Any initial condition that does not lie on the first eigenvector will end up parallel to this, because the exponential for the second will dominate as time gets large. Thus two initial conditions that would work are $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (in the latter case the trajectory lies on the line).

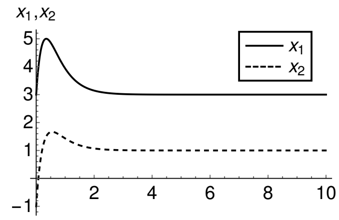
3. [14 points] Consider the systems of equations below. In each the vector $\mathbf{x}$ has components x_1 and x_2.

A. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x}$

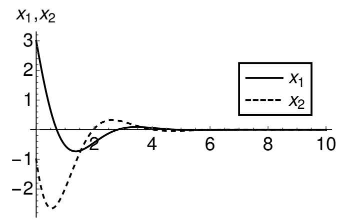
B. $\mathbf{x}' = \begin{pmatrix} 2 & -8 \\ 3 & -8 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Match each of these to one of the graphs to the right (note that two of these are component plots and two are phase portraits). Briefly explain how you know that your matching is correct.

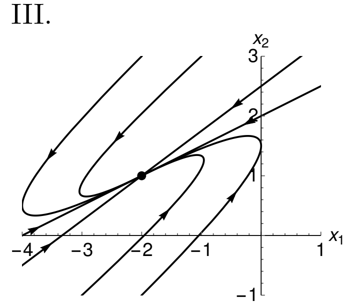
I.



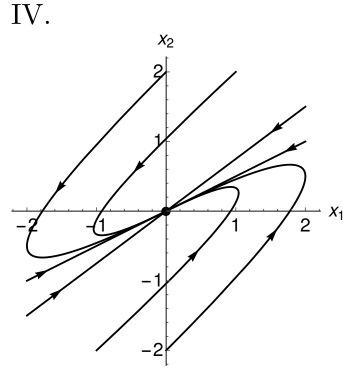
II.



III.



IV.



Solution: It is convenient to start by finding the eigenvalues of the coefficient matrix.

$$\det\left(\begin{pmatrix} 2-\lambda & -8 \\ 3 & -8-\lambda \end{pmatrix}\right) = \lambda^2 + 6\lambda + 8$$
$$= (\lambda + 2)(\lambda + 4) = 0,$$

so that $\lambda = -2$ and $\lambda = -4$. Thus in the phase plane, solutions for A must have two straight line trajectories through the origin (which could be graph IV, but not III); and component plots must go to $x_1 = 0 = x_2$ without oscillation (which excludes plots I and II). Thus A must match graph IV.

Next, B is the same system as A, but with forcing. The equilibrium solution for B is given by

$$2x_1 - 8x_2 = -2, 3x_1 - 8x_2 = 1,$$

so that, subtracting the first from the second, we have $x_1 = 3$. Plugging back into either equation, $x_2 = 1$. Therefore, one solution to B is $x_1 = 3$, $x_2 = 1$; this excludes graph III. For the same reason as for A, the component plots should not be oscillatory, so B must match graph I.

4. [15 points] For each of the following, find explicit, real-valued solutions, as indicated.

a. [7 points] Find the general solution to $y' = 7 - \dfrac{\cos(t)}{2+\sin(t)}y$.

Solution: This is linear, but not separable. Rewriting in standard form, we have $y' + \frac{\cos(t)}{2+\sin(t)}\, y = -7$, so that an integrating factor is $\mu = \exp(\int \frac{\cos(t)}{2+\sin(t)}\, dt) = \exp(\ln|2+\sin(t)|) = 2+\sin(t)$. Then $((2+\sin(t))\, y)' = 7(2+\sin(t))$, so that $(2+\sin(t))y = 7(2t-\cos(t)) + C$, and

$$y = \frac{14t - 7\cos(t)}{2+\sin(t)} + \frac{C}{2+\sin(t)}.$$

b. [8 points] Solve the initial value problem: $\dfrac{y}{t^2+5}\, y' = 1$, $y(0) = -2$

Solution: This is nonlinear, and fortunately separable. Separating, we have $yy' = t^2+5$, so that $\frac{1}{2}\, y^2 = \frac{1}{3}\, t^3 + 5t + \tilde{C}$. Solving for y (and letting $C = 2\tilde{C}$),

$$y = \pm\sqrt{\frac{2}{3}\, t^3 + 10\, t + C}.$$

Applying the initial condition, we must take the negative square root and $C = 4$, so that

$$y = -\sqrt{\frac{2}{3}\, t^3 + 10\, t + 4}.$$

5. [14 points] The volumetric rate at which liquid leaves a cylindrical tank through a circular hole in its bottom is proportional to the square root of the volume of liquid in the tank. Suppose we have a cylindrical tank 5 meters tall with a 1 meter radius (so that its volume is 5π m^3) that is initially full of some liquid. At time $t = 0$, a circular hole opens in the base and more liquid is added at a rate of 1 m^2/hr.

a. [5 points] Write an initial value problem for the volume V of liquid in the tank. (Your answer will involve a constant of proportionality k.) Can you solve your equation? Explain. *(Do not actually solve the equation.)*

Solution: We have $\frac{dV}{dt}$ = (rate in) − (rate out) $= 1 - k\sqrt{V}$, with $V(0) = 5\pi$. This is nonlinear and separable (and thus solvable), but the resulting integral is not one that we are able to evaluate in closed form.

b. [5 points] Suppose that the solution to your equation in (a) is some function $V(t)$. If the liquid in the tank initially contains a particulate at a concentration of 1 g/m^3 and the liquid entering has a particulate concentration of 2 g/m^3, write an initial value problem for the amount of particulate in the tank. (Your answer will involve the unknown function $V(t)$.) Can you solve this equation? Explain. *(Do not actually solve the equation.)*

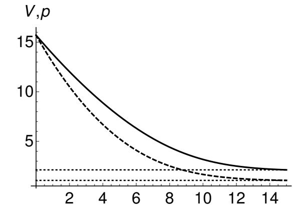
Solution: We can write an equation by considering the tank as a compartment: then $\frac{dp}{dt}$ = (rate in) − (rate out). The rate in is (1 m^3/hr)(2 g/m^3) = 2 (g/hr). The rate out is $(k\sqrt{V})(p(t)/V) = kp/\sqrt{V}$. Thus our differential equation is

$$\frac{dp}{dt} = 2 - \frac{k}{\sqrt{V}}\,p,$$

with initial condition $p(0) = 5\pi$. Assuming that we know the function $V(t)$, this is a linear (but not separable) equation, which we could solve with an integrating factor.

Problem 5, continued.

c. [4 points] What do you expect the long-term value for the volume $V(t)$ to be? Can you predict the long-term value for $p(t)$? If $k = 1$, which of the graphed functions to the right is $V(t)$ and which is $p(t)$? Why?



Solution: Note that the volume starts larger than 1, so initially $V' < 0$. There is an equilibrium volume, $k\sqrt{V} = 1$, so that $V = 1/k^2$, to which the volume will exponentially approach. In the long-term, then, we expect $V \to 1/k^2$, and therefore we expect the long-term dynamics of p to be given by $p' = f(p) = 2 - kp/\sqrt{1/k^2} = 2 - k^3 p$. The equilibrium value is $p_0 = 2/k^3$, and because $f'(p_0) = -k^3 < 0$, it is stable. We therefore expect that $p \to 2/k^3$. Finally, if $k = 1$, then $V \to 1$ and $p \to 2$, so the solid curve in the figure must be $p(t)$ and the dashed one $V(t)$.

6. [10 points] Consider the initial value problem $(1 - y^3)\frac{dy}{dt} = 1$, $y(0) = 0$.

a. [5 points] Without solving it, will this initial value problem have a unique solution?

Solution: Note that this is the same as $y' = f(y) = 1/(1 - y^3)$. The function f and its derivative f_y are discontinuous only if $y = 1$, so there will be a unique solution provided $y(0) \neq -1$. Thus we are guaranteed that there will be a unique solution.

b. [5 points] Solve the problem. Based on your solution, for what range of t and y values would you expect the solution to exist? Why?

Solution: Integrating both sides with respect to t, we have $y - \frac{1}{4}y^4 + C = t$, so that with $y(0) = 0$ we require $C = 0$ and have $t = y - \frac{1}{4}y^4$. Thinking of t as a function of y, this is a curve like a parabola opening down around the t-axis. Its vertex is when $\frac{d}{dy}(y - \frac{1}{4}y^4) = 1 - y^3 = 0$, or $y = 1$. When $y = 1$, $t = \frac{3}{4}$, and at that point the solution vanishes. Thus we have a (unique) solution only for $0 \le t < \frac{3}{4}$, $0 \le y < 1$. We may include $t = \frac{3}{4}$ and $y = 1$ in these, as the function continues to be well defined there.

7. [16 points] In lab we considered the van der Pol system $x' = y$, $y' = -x - \mu y \frac{df}{dx}$. Here, we suppose that $f'(x) = |x| - a$, so that this becomes $x' = y$, $y' = -x - \mu y\,(|x| - a)$.

a. [3 points] Find the critical point for this system.

Solution: Critical points are where $x' = y' = 0$, so from the first equation, $y = 0$. Then if $y = 0$ the second equation is $0 = -x$, so $x = 0$ also. The only critical point is $(0, 0)$.

b. [3 points] Linearize the system at your critical point from (a).

Solution: The only nonlinear term in the system is $y|x|$, which for x and y near $(0, 0)$ will be very small; thus, our linearization is

$$x' = y, \quad y' = -x + a\mu y.$$

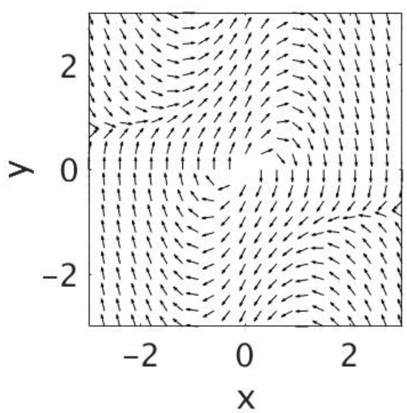
c. [5 points] Suppose that your linear system from (b) is, for some k that depends on both of μ and a, $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix} \mathbf{x}$. Determine the type and stability of the critical point.

Solution: The coefficient matrix of the nonlinear system is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & k \end{pmatrix}$, so eigenvalues are given by $\det(\begin{pmatrix} -\lambda & 1 \\ -1 & k - \lambda \end{pmatrix}) = \lambda^2 - k\lambda + 1 = 0$. Thus, $\lambda = \frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 - 4}$. This will have a positive real part if $k > 0$, so for $k > 0$ the origin is unstable while for $k < 0$ it is asymptotically stable. For $|k| < 2$, λ will be complex valued and we will have a spiral point; for $|k| > 2$, it will be a node. If $k = \pm 2$, we have a repeated eigenvalue and therefore the degenerate case with a single eigenvector.

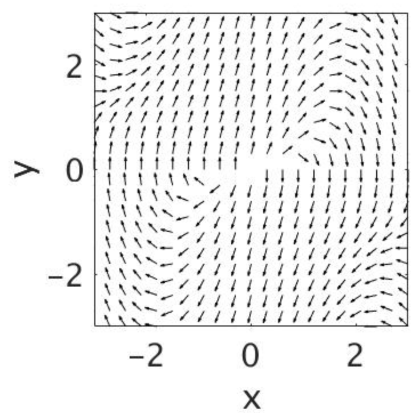
Problem 7, continued. We are considering the system $x' = y$, $y' = -x - \mu y\,(|x| - a)$

d. [5 points] Each of the direction fields below is generated for the nonlinear equation we are considering, picking μ and a so that the value of k given in (c) is one of -2, -1, 1, or 2. Identify, with a short explanation, which graph corresponds to which value of k.

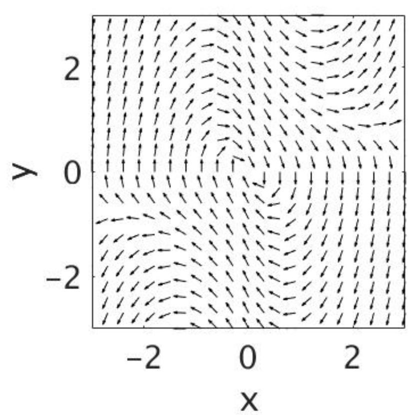
i.



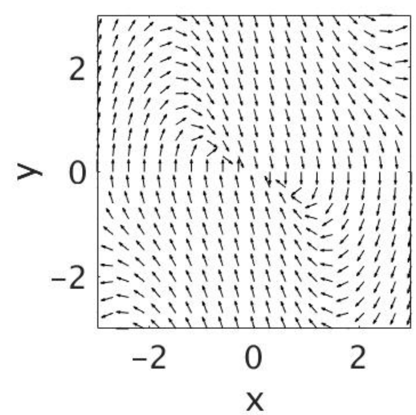
ii.



iii.



iv.



Solution: For $k = -2$ and $k = -1$ trajectories must approach the origin, so these are iii and iv. For $k = -1$ the trajectories must be a spiral, which is iii. Thus iii corresponds to $k = -1$ and iv to $k = -2$. Then if $k = 1$ we must have spirals out from the origin, which matches i; and for $k = 2$, one straight line solution, which matches ii.