# Math 216 — Second Midterm

16 November, 2017

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This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.



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**1**. [15 points] For each of the following, find explicit real-valued solutions as indicated.

**a**. [7 points] Find the solution to the initial value problem $y'' + 2y' + 17y = 0$, $y(0) = 1$, $y'(0) = 0$.

*Solution:* The solution will look like $y = e^{\lambda t}$. Plugging in, we have the characteristic equation $\lambda^2 + 2\lambda + 17 = (\lambda + 1)^2 + 16 = 0$. Thus $\lambda = -1 \pm 4i$. A general solution is $y = c_1 e^{-t}\cos(4t) + c_2 e^{-t}\sin(4t)$. Applying the initial conditions, we have $c_1 = 1$, and $-1 + 4c_2 = 0$, so that $c_2 = \frac{1}{4}$. Thus the solution is

$$y = e^{-t}\cos(4t) + \frac{1}{4}\,e^{-t}\sin(4t).$$

We can, of course, also solve this with Laplace transforms. The forward transform gives, with $Y = \mathcal{L}\{y\}$, $s^2Y - s + 2sY + 17Y = 0$, so that $Y = \frac{s}{s^2+2s+17} = \frac{(s+1)-1}{(s+1)^2+16} = \frac{s+1}{(s+1)^2+16} - \frac{1}{(s+1)^2+16}$. The two terms invert using the rules for cosine, sine, and $F(s-c)$ to give the result above.

**b**. [8 points] Find the general solution to the equation $y'' + 5y' + 4y = e^t + 8t$.

*Solution:* The complementary homogeneous solution will be a linear combination of exponentials $y = e^{\lambda t}$. Plugging in, we have $\lambda^2 + 5\lambda + 4 = (\lambda + 4)(\lambda + 1) = 0$, so $\lambda = -4$ or $\lambda = -1$ and $y_c = c_1 e^{-4t} + c_2 e^{-t}$.

To find the particular solution, consider the exponential and linear term separately. For the former, we guess $y_{p1} = ae^t$, so that $(1 + 5 + 4)ae^t = 10e^t$, and $a = \frac{1}{10}$. For the latter, we guess $y_{p2} = a_0 + a_1 t$, so that $5a_1 + 4a_0 + 4a_1 t = 8t$, and $a_1 = 2$, $a_0 = -\frac{5}{2}$.

Thus the general solution is

$$y = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{10}\,e^t - \frac{5}{2} + 2t.$$

**2**. [12 points] The following problems consider a non-homogeneous second-order linear differential equation $L[y] = g(t)$. Suppose that $y_1$ and $y_2$ are solutions to this equation, and that $y_3$, $y_4$, and $y_5$ are solutions to the complementary homogeneous problem $L[y] = 0$.

**a**. [3 points] Can you say what problem each of the following solve? If so, indicate what it is; if not, write "none." (No explanation necessary.)

i. $y_1 - y_2$

ii. $y_1 - y_3$

iii. $y_1 + y_2 + y_3 + y_4 + y_5$

*Solution:* i. $L[y_1 - y_2] = 0$; ii. $L[y_1 - y_3] = g(t)$; iii. $L[y_1 + y_2 + y_3 + y_4 + y_5] = 2g(t)$.

**b**. [3 points] Explain how you are able to determine your answers in (a), or why it is not possible to tell.

*Solution:* All of these stem from the linearity of the operator $L$, which means that $L[ay + bz] = aL[y] + bL[z]$. Thus we have:
i. $L[y_1 - y_2] = L[y_1] - L[y_2] = g(t) - g(t) = 0$,
ii. $L[y_1 - y_3] = L[y_1] - L[y_3] = g(t) - 0 = g(t)$, and
iii. $L[y_1 + y_2 + y_3 + y_4 + y_5] = L[y_1] + L[y_2] + L[y_3] + L[y_4] + L[y_5] = g(t) + g(t) + 0 + 0 + 0 = 2g(t)$.

**c**. [6 points] The following statements are not guaranteed to be true. Explain why.

i. The solution to the initial problem $L[y] = 0$, $y(0) = y_0$, $y'(0) = v_0$ (for any $y_0$ and $v_0$) can be written as $y = c_1y_3 + c_2y_4 + c_3y_5$ for some $c_1$, $c_2$, and/or $c_3$ (where one or more of $c_1$, $c_2$, and $c_3$ may be zero).

ii. Because both $y_1$ and $y_2$ satisfy $L[y] = g(t)$, we must have $y_1 = y_2$.

*Solution:* i. This is false because we don't know that there are two linearly independent solutions from the three given. In order to be able to solve the initial value problem for any initial conditions we must start with a general solution, which requires that we have linearly independent solutions. The problem doesn't exclude the possibility that $y_3 = y_4 = y_5 = 0$, for example.
ii. This is false because $y_1$ and $y_2$ are only specified up to an additive multiple of a homogeneous solution: for example, $y_2$ could be $y_1 + y_3$; then $L[y_1 + y_3] = L[y_1] + L[y_3] = g(t) + 0$, but if $y_3 \neq 0$ the functions $y_1$ and $y_2$ are not equal.

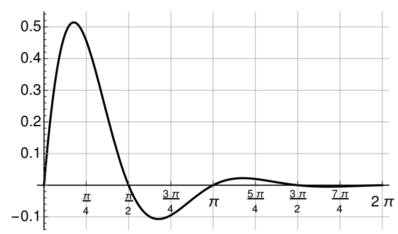
**3.** [15 points] For all of the following, the equations are linear, constant-coefficient, and second-order, with the coefficient of $y''$ picked to be one.

**a.** [5 points] If the differential equation is nonhomogeneous and the general solution is $y = c_1 e^{-2t} + c_2 e^{-3t} + 4\cos(2t)$, what is the differential equation?

*Solution:* The characteristic equation of the homogeneous problem must have roots $-2$ and $-3$, so it is $(\lambda+2)(\lambda+3) = \lambda^2 + 5\lambda + 6$. The differential equation is therefore $y'' + 5y' + 6y = g(t)$. The particular solution is $y_p = 4\cos(2t)$; plugging this in, we have $-16\cos(2t) - 10\sin(2t) + 24\cos(2t) = g(t)$, so $g(t) = 8\cos(2t) - 40\sin(2t)$, and the equation is

$$y'' + 5y' + 6y = 8\cos(2t) - 40\sin(2t)).$$

**b.** [5 points] If the graph to the right shows the movement of a unit mass on a spring with damping constant 2, set in motion with an initial velocity of 1 m/s, write an initial value problem modeling the position of the mass.




*Solution:* From the graph we see that $y(0) = 0$ and from the problem statement we know that $y'(0) = 1$. The solution is in the form of a decaying exponential, so the homogeneous solutions to the problem are of the form $e^{-at}\cos(bt)$ and $e^{-at}\sin(bt)$. The period of the oscillation is $\pi$, so $b = 2$. Then we know that $y'' + 2y' + ky = 0$, so that the characteristic equation is $\lambda^2 + 2\lambda + k = (\lambda+1)^2 + k - 1 = 0$, which has roots $\lambda = -1 \pm i\sqrt{k-1}$. From the form of the solutions, we know $\sqrt{k-1} = 2$, so $k = 5$. Thus the initial value problem is

$$y'' + 2y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

**c.** [5 points] Consider the (linear, constant-coefficient. . . ) equation $L[y] = 0$ and the equivalent system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. If one solution to the equation $L[y] = 0$ is $y = e^{-t}$, what is a corresponding solution to the system? If the coefficient of $y$ in the equation is 3, what is the differential equation?

*Solution:* If $y = e^{-t}$ is a solution to $L[y] = 0$, then $\mathbf{x} = \begin{pmatrix} e^{-t} & -e^{-t} \end{pmatrix}^T$ is a solution to the system, so that $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = -1$. Because the system is derived from a second order equation, and given the coefficient of $y$ is 3, we know that $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -3 & k \end{pmatrix}$. Then, using $\mathbf{v}$ and $\lambda = -1$, we have $\begin{pmatrix} 1 & 1 \\ -3 & k+1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \mathbf{0}$, so $k = -4$ and the equation is

$$y'' + 4y' + 3y = 0.$$

Alternately, we know $y'' + ay' + 3y = 0$. Thus $\lambda^2 + a\lambda + 3 = (\lambda+1)(\lambda+r) = 0$, for some $r$. Expanding the right-hand side and matching powers of $\lambda$, $r + 1 = a$ and $r = 3$. Thus $a = 4$, and we obtain the equation above.

**4**. [14 points] Recall that the nonlinear model for the number of photons $P$ and population inversion $N$ in a ruby laser that we considered in lab 3 had an equilibrium point $(P, N) = (1, A-1)$. If we assume that $A = A_0 + a\cos(\omega t)$, with $a$ a very small value, the dynamics of the system near the equilibrium point are modeled by the linear system $u' = -\gamma(Au + v) + \gamma a \cos(\omega t)$, $v' = (A-1)u$

**a**. [4 points] Rewrite this linear system as a second order equation in $v$.

*Solution:* Note that we have $u = \frac{1}{A-1} v'$ Plugging this into the first equation, we get $\frac{1}{A-1} v'' + \frac{\gamma A}{A-1} v' + \gamma v = \gamma a \cos(\omega t)$. Multiplying through by $A-1$, we obtain

$$v'' + \gamma A v' + \gamma(A-1)v = \gamma(A-1)a\cos(\omega t).$$

**b**. [6 points] Suppose that the second order equation you obtain in (a) is $v''+2v'+v = \cos(\omega t)$, so that the solution to the complementary homogeneous problem is $v_c = c_1 e^{-t} + c_2 te^{-t}$. Set up the solution for $v_p$ using variation of parameters, and solve them to obtain explicit equations for $u_1'$ and $u_2'$ in terms of $t$ only. *(Do not solve these to find $u_1$ and $u_2$.)*

*Solution:* Our guess for $v_p$ is $v_p = u_1 e^{-t} + u_2 te^{-t}$. We know with variation of parameters, the equations that $u_1$ and $u_2$ must satisfy are $u_1' e^{-t} + u_2' te^{-t} = 0$, $-u_1' e^{-t} + u_2'(e^{-t} - te^{-t}) = \cos(\omega t)$. The first give $u_1' = -tu_2'$, so the second becomes $u_2' = e^t \cos(\omega t)$. Therefore, the equation for $u_1$ becomes $u_1' = -te^t \cos(\omega t)$.

**c**. [4 points] It turns out that, for some $A$ and $B$, $v_p = A\cos(\omega t) + B\sin(\omega t)$. Representative values of $A$ and $B$ are given for different values of $\omega$ in the table below. Does the system exhibit resonance? Write the response $v_p$ to a forcing of $\cos(2t)$ in phase-amplitude form.

| $\omega =$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $A =$ | 0 | $-0.12$ | $-0.08$ | $-0.06$ | $-0.04$ |
| $B =$ | 1 | 0.64 | 0.36 | 0.22 | 0.15 |

*Solution:* We see that the amplitude of $v_p$, $R = \sqrt{A^2 + B^2}$, is never larger than one, so there is no resonance. When $\omega = 2$, $A = -0.12$ and $B = 0.64$, so $R = \sqrt{0.144 + 0.4096} \approx \sqrt{0.424}$. The phase shift $\delta$ has $A < 0$ and $B > 0$, so we need $\delta = \pi - \arctan(0.64/0.12) \approx \pi - \arctan(5.33)$. Thus $v_p = \sqrt{0.424}\cos(2t - (\pi - \arctan(5.33)))$.

**5**. [14 points] Consider the initial value problem $y'' + 4y' + 4y = 9e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

**a**. [6 points] Solve the problem *without* using Laplace transforms.

*Solution:* We first note that with $y = e^{\lambda t}$, the characteristic equation for the complementary homogeneous solution is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$. Thus $\lambda = -2$, repeated, and $y_c = c_1 e^{-2t} + c_2 t e^{-2t}$.
For the particular solution, we therefore guess $y_p = at^2t^{-2t}$, so that $y_p' = 2ate^{-2t} - 2at^2e^{-2t}$, and $y_p'' = 2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t}$. Plugging in, we have

$$2ae^{-2t} - 8ate^{-2t} + 4at^2e^{-2t} + 4(2ate^{-2t} - 2at^2e^{-2t}) + 4at^2e^{-2t} = 9e^{-2t},$$

so that $a = \frac{9}{2}$. Thus the general solution is

$$y = c_1e^{-2t} + c_2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

Applying the initial conditions, $y(0) = c_1 = 1$, and $y'(0) = -2c_1 + c_2 = -2 + c_2 = 0$, so that $c_2 = 2$. The solution is therefore

$$y = e^{-2t} + 2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

**b**. [8 points] Solve the problem *with* Laplace transforms.

*Solution:* We transform both sides of the equation to find, with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s + 4sY - 4 + 4Y = \frac{9}{s+2},$$

so that $Y = \frac{s+4}{(s+2)^2} + \frac{9}{(s+2)^3}$. The easiest way to find the inverse transform is to rewrite $Y$ solely in terms of $s + 2$:

$$Y = \frac{(s+2)+2}{(s+2)^2} + \frac{9}{(s+2)^3} = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{9}{(s+2)^3},$$

so that (applying for the second and third the rules for $F(s - c)$ and $1/s^n$)

$$y = e^{-2t} + 2te^{-2t} + \frac{9}{2}t^2e^{-2t}.$$

We can also find the inverse transform of the first by using partial fractions and letting $\frac{s+4}{(s+2)^2} = \frac{A_0}{s+2} + \frac{A_1}{(s+2)^2}$. Clearing the denominator, $s + 4 = A_0(s + 2) + A_1$, so that $A_0 = 1$ and $A_1 = 2$. Thus

$$Y = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{9}{(s+2)^3},$$

and the inverse transform is as before.

**6**. [15 points] Complete each of the following problems having to do with the Laplace transform.

**a**. [5 points] Find the inverse Laplace transform of $F(s) = \dfrac{5s}{s^2+4s+6}$

*Solution:* Note that $s^2+4s+6 = (s+2)^2+2$. Thus $\frac{5s}{s^2+4s+6} = \frac{5(s+2)}{(s+2)^2+2} - \frac{10}{(s+2)^2+2}$, and the inverse transform is

$$\mathcal{L}^{-1}\{\frac{5(s+2)}{(s+2)^2+2} - \frac{10}{(s+2)^2+2}\} = 5e^{-2t}\cos(\sqrt{2}\,t) - 5\sqrt{2}\,e^{-2t}\sin(\sqrt{2}\,t).$$

**b**. [5 points] Given that $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to derive the transform rule $-F'(s) = \mathcal{L}\{tf(t)\}$.

*Solution:* We have $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Thus, assuming that the integral converges in a manner that allows differentiation through the integral sign,

$$\begin{aligned} -F'(s) = -\int_0^\infty \frac{d}{ds}(e^{-st}) f(t)\,dt &= -\int_0^\infty -te^{-st} f(t)\,dt \\ &= \int_0^\infty e^{-st} t f(t)\,dt = \mathcal{L}\{tf(t)\}. \end{aligned}$$

**c**. [5 points] Consider the initial value problem $ty''+y=0$, $y(0)=1$, $y'(0)=0$. If $Y = \mathcal{L}\{y\}$, what equation does $Y$ satisfy?

*Solution:* Note that $\mathcal{L}\{y''\} = s^2Y - s$, and that the transform rule in part (b) above (also, rule B on the formula sheet) gives $\mathcal{L}\{tf(t)\} = -F'(s)$. Thus, with $f(t) = y''$ and $F(s) = s^2Y(s) - 1$, we have $\mathcal{L}\{ty''\} = -s^2Y'(s) - 2sY(s) + 1$. Transforming the differential equation, we therefore have

$$-s^2Y'(s) - 2sY(s) + 1 + Y(s) = 0,$$

a first-order equation for $Y(s)$. Note that we can rewrite this as $Y' + \frac{2s-1}{s^2}Y = \frac{1}{s^2}$, so the equation is first-order and linear, and therefore solvable at least in principle. It is not clear if we would be able to invert the resulting $Y(s)$, however.

**7.** [15 points] Consider the system of differential equations $x' = 3x + 4y$, $y' = 2x + y$, with initial conditions $x(0) = 0$, $y(0) = 2$.

**a.** [6 points] Using Laplace transforms, find explicit equations for $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

*Solution:* We have $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$. Then, transforming both equations, we have $sX = 3X + 4Y$ and $sY - 2 = 2X + Y$. Thus $Y = \frac{1}{4}(s-3)X$, and the second equation becomes $(s-1)\frac{1}{4}(s-3)X - 2X = 2$, or $X = \frac{8}{s^2-4s-5} = \frac{8}{(s-5)(s+1)}$. Plugging back into the equation for $Y$, we have

$$X = \frac{8}{(s-5)(s+1)}, \quad Y = \frac{2(s-3)}{(s-5)(s+1)}.$$

**b.** [4 points] Find $x$ and $y$ in terms of any constants you may have in partial fractions expansions of $X$ and $Y$ (that is, do not solve for the values of those constants).

*Solution:* We can find the inverse transforms for both of these by using partial fractions: for $X$, we have $X = A/(s-5) + B/(s+1)$. For $Y$, we have $Y = C/(s-5) + D/(s+1)$. Thus

$$x = Ae^{5t} + Be^{-t}, \quad y = Ce^{5t} + De^{-t}.$$

If we complete the solution, which is not required in the problem, we have in the partial fractions decomposition for $X$ $8 = A(s+1) + B(s-5)$. Plugging in $s = 5$ and $s = -1$, we get $A = \frac{4}{3}$ and $B = -\frac{4}{3}$. Similarly for $Y$, we have $2s - 6 = C(s+1) + D(s-5)$. Plugging in $s = 5$ and $s = -1$, we have $C = \frac{2}{3}$ and $D = \frac{4}{3}$. Thus the wolution is

$$x = \frac{4}{3}e^{5t} - \frac{4}{3}e^{-t}, \quad y = \frac{2}{3}e^{5t} + \frac{4}{3}e^{-t}.$$

**c.** [5 points] If we rewrote the system as a second order differential equation $L[y] = 0$ for $y$, what would the characteristic equation for $\lambda$ be? What is the linear operator $L$?

*Solution:* The characteristic equation for $\lambda$ can be seen from the denominator of $X$ and $Y$, or from the exponentials in $x$ and $y$: $(\lambda - 5)(\lambda + 1) = \lambda^2 - 4\lambda - 5 = 0$. Thus the operator $L$ is $L = D^2 - 4D - 5$, or some constant multiple thereof.