Math 216 — Final Exam

14 December, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Solve the initial value problem $2ty' - y = -3t^2$, $y(1) = 2$. (Consider $t \geq 1$.)

Solution: This is linear but not separable. In standard form, we have

$$y' - \frac{1}{2t}\,y = -\frac{3}{2}\,t,$$

so that an integrating factor is $\exp(\int -\frac{1}{2t}\,dt) = -\frac{1}{2}\exp(\ln(t)) = t^{-1/2}$. Multiplying through by this, we have $(t^{-1/2}y)' = -\frac{3}{2}\,t^{1/2}$. Integrating and solving for y, we have

$$y = -t^2 + Ct^{1/2}.$$

The initial condition requires that $2 = -1 + C$, and $C = 3$.

b. [8 points] Find the general solution to $y'' - 4y = 12t + e^{-2t}$.

Solution: We know the general solution will be $y = y_c + y_p$, where y_c is the solution to the complementary homogeneous problem and y_p a particular solution. For y_c we take $y = e^{\lambda t}$, so that $\lambda^2 - 4 = 0$, and $\lambda = \pm 2$. Thus $y_c = c_1 e^{-2t} + c_2 e^{2t}$.
Then for the particular solution we consider the terms $12t$ and e^{-2t} separately. For $12t$, we guess $x_p = at + b$, so that $-4at - 4b = 12t$, and $b = 0$, $a = -3$.
For e^{-2t}, we guess $y_p = ate^{-2t}$, because e^{-2t} appears in the complementary homogeneous solution. Then $y_p' = ae^{-2t} - 2ate^{-2t}$, and $y_p'' = -4ae^{-2t} + 4ate^{-2t}$. Plugging in, we have

$$a\left((-4e^{-2t} + 4te^{-2t}) - 4te^{-2t}\right) = -4ae^{-2t} = e^{-2t},$$

so that $a = -\frac{1}{4}$.
The general solution is then

$$y = c_1 e^{-2t} + c_2 e^{2t} - 3t - \frac{1}{4}\,t\,e^{-2t}.$$

2. [15 points] For each of the following, find explicit real-valued solutions as indicated.

a. [7 points] Find the general solution to the system $x' = 2x + 3y$, $y' = 5x + 4y$.

Solution: Note that the coefficient matrix for the system in vector form is $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$.
The eigenvalues satisfy $(2-\lambda)(4-\lambda) - 15 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1) = 0$, so $\lambda = 7$ or $\lambda = -1$.
If $\lambda = 7$, the eigenvector satisfies $\begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. If $\lambda = -1$, we have $\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix} \mathbf{v} = \mathbf{0}$, and $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{7t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} 3c_1 e^{7t} + c_2 e^{-t} \\ 5c_1 e^{7t} - c_2 e^{-t} \end{pmatrix}.$$

b. [8 points] Solve the initial value problem $y'' - 4y' + 13y = -\delta(t-4)$, $y(0) = 1$, $y'(0) = 3$.

Solution: Because of the delta function, we will use Laplace transforms. Transforming both sides and letting $Y = \mathcal{L}\{y\}$, we have

$$(s^2 - 4s + 13)Y - s - 3 + 4 = -e^{-4s},$$

so that

$$Y = -\frac{e^{-4s}}{s^2 - 4s + 13} + \frac{s-1}{s^2 - 4s + 13}.$$

To invert the transform, we rewrite the right-hand side as

$$Y = -\frac{e^{-4s}}{(s-2)^2 + 9} + \frac{s-2}{(s-2)^2 + 9} + \frac{1}{(s-2)^2 + 9}.$$

We can invert this by inspection to obtain:

$$y = -\frac{1}{3}\, u_4(t)\, e^{2(t-4)} \sin(3(t-4)) + e^{2t}\cos(3t) + \frac{1}{3}\, e^{2t} \sin(3t).$$

3. [12 points] For parts (a) and (b), identify each as true or false, and give a short mathematical calculation, with explanation, justifying your answer. For part (c) the statement is false. Explain why.

a. [4 points] If L is a linear second-order differential operator, y_1 and y_2 are non-zero functions for which $L[y_1] = L[y_2] = 0$, and y_3 is a function for which $L[y_3] = \frac{3}{2+t^2}$, then for any c_1, c_2, and c_3, $y = c_1y_1 + c_2y_2 + c_3y_3$ solves $L[y] = \frac{3}{2+t^2}$.

True $\boxed{\text{False}}$

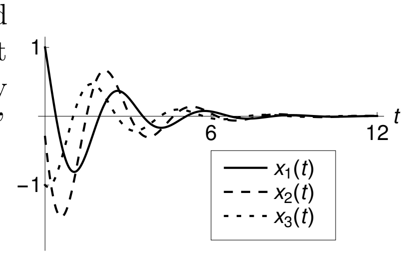
Solution: Because L is linear, we have $L[c_1y_1+c_2y_2+c_3y_3] = c_1L[y_1]+c_2L[y_2]+c_3L[y_3] = (c_1 + c_2)(0) + c_3(\frac{3}{2+t^2}) \neq \frac{3}{2+t^2}$ (unless $c_3 = 1$).

b. [4 points] If L is a linear second-order differential operator with continuous coefficients, and y_1 and y_2 are non-zero functions satisfying $L[y] = 0$, $y_1(0) = y_2'(0) = 0$ and $y_1'(0) = y_2(0) = 1$, then a general solution to $L[y] = 0$ is given by $y = c_1y_1 + c_2y_2$.

$\boxed{\text{True}}$ False

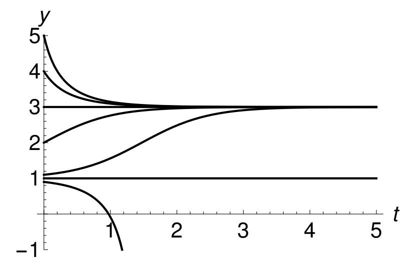
Solution: Note that the Wronskian of y_1 and y_2, at $t = 0$, is $W[y_1, y_2](0) = y_1(0)y_2'(0) - y_1'(0)y_2(0) = -1 \neq 0$. Thus y_1 and y_2 are linearly independent at zero (and hence everywhere), and a general solution is given by $y = c_1y_1 + c_2y_2$.

c. [4 points] Suppose $\mathbf{A}$ is a real-valued 3×3 matrix, and that the three curves shown to the right are the component plots of a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$, as indicated. Explain why the statement "eigenvalues of $\mathbf{A}$ must be complex-valued" is false.



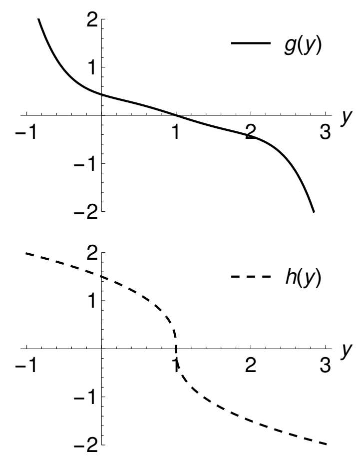
Solution: If $\mathbf{A}$ is a real-valued 3×3 matrix, complex-valued eigenvalues must come in complex-conjugate pairs, and there are three eigenvalues. Here we see decaying oscillatory behavior, so there must be a (pair of complex-conjugate) eigenvalue(s), and one real one.

4. [6 points] Representative solution curves for a first-order differential equation $y' = f(y)$ are shown in the figure to the right. Write a possible function $f(y)$ that could give this behavior. Explain why your function could be correct.



Solution: We note that the differential equation has equilibrium solutions at $y = 1$ and $y = 3$, and $y' = f(y) < 0$ for $y > 3$. Thus one possible function is $f(y) = -(y-3)(y-1)$.

5. [6 points] Consider the two equations
A. $y' = g(y)$ and
B. $y' = h(y)$,
where $g(y)$ and $h(y)$ are given in the figures to the right. Which of the following three statements must hold for each equation?
1. "The equation with the initial condition $y(0) = 1$ must have a unique solution."
2. "The equation with an initial condition $y(0) = y_0 < 1$ must have a unique solution."
3. "The solution to the equation with an initial condition $y(0) = y_0 < 1$ will asymptotically approach $y = 1$ as $t \to \infty$."
Briefly explain your answers.



Solution: We note that $g(y)$, $g'(y)$, $h(y)$, and $h'(y)$ appear continuous everywhere but at $y = 1$, and $g(y)$ and $g'(y)$ are also continuous at $y = 1$ while $h'(y)$ appears to be undefined there. Thus the existence and uniqueness theorem will hold for equation A ($y' = g(y)$), which guarantees that all of statements 1, 2, and 3 must be true (if we start below $y = 1$ solutions will increase, but $y = 1$ is a solution, so any solution starting below $y = 1$ cannot reach $y = 1$ or solution curves would intersect).
For equation B ($y' = h(y)$), however, we are not guaranteed a unique solution when $y(0) = 1$. Thus statement 1 is not guaranteed to be true, though statement 2 is. And for the same reason we cannot guarantee that an initial condition below $y = 1$ will not end up intersecting at $y = 1$. Thus statement 3 is not guaranteed to be true either.

6. [11 points] For parts (a) and (b) explain what is wrong with the calculation. For part (c), show the indicated result.

a. [4 points] *(Explain why this is incorrect:)* $\mathcal{L}^{-1}\{\frac{4e^{-3s}}{(s^2+4)}\} = 2u_3(t)\,\sin(2t)$

Solution: The rule for the exponential factor is $\mathcal{L}^{-1}\{e^{-sc}F(s)\} = u_c(t)f(t-c)$. Here the function $\sin(2t)$ hasn't been translated as it should.

b. [4 points] *(Explain why this is incorrect:)* If $f(t) = \begin{cases} 2t, & t \leq 3 \\ 5, & t > 3 \end{cases}$, then $\mathcal{L}\{f(t)\} = \frac{2}{s^2} + \frac{5e^{-3s}}{s}$.

Solution: Note that the inverse transform of the result given is $\mathcal{L}^{-1}\{\frac{2}{s^2} + \frac{5e^{-3s}}{s}\} = 2t + 5u_3(t) = \begin{cases} 2t, & t < 3 \\ 2t+5, & t \geq 3 \end{cases}$: instead of turning off the $2t$ and turning on the constant function 5, we've just added 5 starting at $t = 3$.

c. [3 points] Use convolution and properties of $\delta(t-c)$ to show that $\mathcal{L}^{-1}\{e^{-sc}F(s)\} = u_c(t)f(t-c)$, given that $\mathcal{L}\{f(t)\} = F(s)$. *(Note: this may be challenging, and you may want to leave it until you've worked other problems on the final.)*

Solution: Note that, formally, $\mathcal{L}^{-1}\{e^{-sc}\} = \delta(t-c)$. Then convolution says

$$L^{-1}\{e^{-sc}F(s)\} = \int_0^t \delta(x-c)f(t-x)\,dt.$$

If $t < c$, the delta function is uniformly zero, and we obtain the inverse transform 0. If $t > c$, the integral of the delta function against $f(t-x)$ filters out the function at the value $x = c$, so that we get $f(t-c)$. Thus

$$L^{-1}\{e^{-sc}F(s)\} = \begin{cases} 0, & t < c \\ f(t-c), & t > c \end{cases} = u_c(t)f(t-c).$$

We note that the value at the discontinuity is ambiguous, which is allowed.

7. [9 points] Consider the system $\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The coefficient matrix has eigenvalues and eigenvectors $\lambda = -2$ and $\lambda = \pm i$, with $\mathbf{v}_{-2} = \begin{pmatrix} -5 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{v}_{\pm i} = \begin{pmatrix} 0 \\ 1 \\ \pm i \end{pmatrix}$.

a. [6 points] Write a general, real-valued solution to the system.

Solution: Separating real and imaginary parts of the complex solutions, we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} -5 \\ 2 \\ 6 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ \cos(t) \\ -\sin(t) \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ \sin(t) \\ \cos(t) \end{pmatrix}.$$

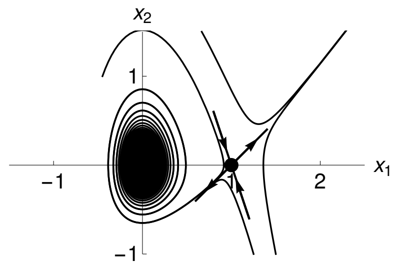
b. [3 points] As $t \to \infty$, what happens to solution trajectories? If we look at trajectories for large times, what will we see?

Solution: As $t \to \infty$, the first term in the general solution will vanish, and we will be left with oscillatory terms in the yz-plane from the last two solutions. Thus for large times, we will see circles in the yz-plane.

8. [12 points] Suppose that for some nonlinear second-order differential equation $y'' = f(y)$ we can write an equivalent system of two first-order differential equations $x_1' = F(x_1, x_2)$, $x_2' = G(x_1, x_2)$. Critical points of the latter are $\mathbf{x}_0 = (0, 0)$ and $\mathbf{x}_1 = (1, 0)$. The Jacobian at these points is $\mathbf{J}(\mathbf{x}_0) = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{J}(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$.

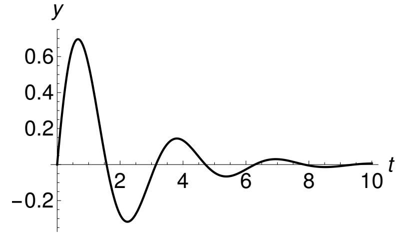
a. [8 points] Sketch a phase portrait for the nonlinear system.

Solution: At $\mathbf{x}_0$ the Jacobian has eigenvalues given by $\lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0$, so that $\lambda = -1 \pm i\sqrt{2}$, and $\mathbf{x}_0$ is a stable spiral point. Note that (considering the point $(1, 0)$, where the derivatives on a trajectory are $(0, -3)$) the spiral moves in a clockwise direction.
At $\mathbf{x}_1$, the Jacobian has eigenvalues given by $\lambda^2 + 2\lambda - 3 = (\lambda + 1)^2 - 4 = 0$, so $\lambda = -3$ or $\lambda = 1$. If $\lambda = -3$, $\mathbf{v} = \begin{pmatrix} 1 & -3 \end{pmatrix}^T$, and if $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.
Plotting these and adding the expected trajectories gives the phase portrait below.



b. [4 points] Based on your phase portrait, sketch a qualitatively accurate graph of y as a function of t if we start with the initial condition $y(0) = 0$, $y'(0) = 1$.

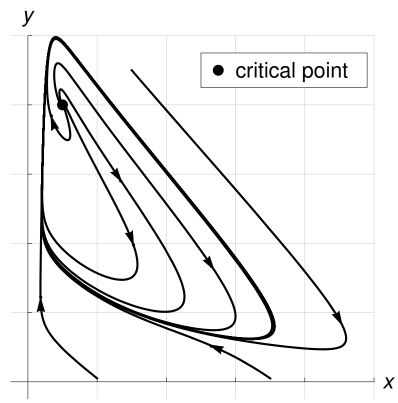
Solution: From the phase portrait, we see that a trajectory starting at $(0, 1)$ should spiral in to the origin. Thus we know that $y(0) = 0$ and $y'(0) = 1$, and that y must be a decaying sinusoidal function. Thus we have a graph like that shown below.



9. [14 points] The Brusselator is a nonlinear model of a chemical reaction which can have oscillatory concentrations x and y of the chemicals in the reaction. A model for this is

$$x' = 1 - (b+1)x + \frac{1}{4}x^2 y, \quad y' = bx - \frac{1}{4}x^2 y.$$

The figure to the right gives the phase portrait for this system for some value of b.



a. [4 points] What are the coordinates of the critical point shown? *(Note that your answer may involve the parameter b.)*

Solution: Critical points are where $x' = 1 - x((b+1) - \frac{1}{4}xy) = 0$ and $y' = x(b - \frac{1}{4}xy) = 0$. The latter says that $x = 0$ or $xy = 4b$. If $x = 0$, the first is unsolvable. If $xy = 4b$, the first becomes $1 - x(b + 1 - b) = 0$, so $x = 1$ and $y = 4b$. The critical point is $(x, y) = (1, 4b)$.

b. [7 points] Given the behavior shown in the phase portrait, what can you say about the parameter b?

Solution: The phase portrait shows that the linear behavior of the critical point is that of an unstable spiral point. To see how b may determine this, we find the Jacobian and investigate the behavior at the critical point. The Jacobian is

$$J = \begin{pmatrix} -(b+1) + \frac{1}{2}xy & \frac{1}{4}x^2 \\ b - \frac{1}{2}xy & -\frac{1}{4}x^2 \end{pmatrix}\Bigg|_{x=1,y=4b} = \begin{pmatrix} b-1 & \frac{1}{4} \\ -b & -\frac{1}{4} \end{pmatrix}.$$

The behavior of the critical point will be determined by the eigenvalues of this Jacobian, which are given by $(b - 1 - \lambda)(-\frac{1}{4} - \lambda) + \frac{1}{4}b = \lambda^2 + (\frac{5}{4} - b)\lambda + \frac{1}{4} = 0$, or

$$\lambda = \frac{1}{2}(b - \frac{5}{4}) \pm \frac{1}{2}\sqrt{(b - \frac{5}{4})^2 - 1}.$$

For this to give an unstable spiral, we must have $b > \frac{5}{4}$, and $(b - \frac{5}{4})^2 - 1 < 0$. The latter says that $-1 < (b - \frac{5}{4}) < 1$, so we know that $\frac{5}{4} < b < \frac{9}{4}$.
It is possible from the picture that we have real eigenvalues, but we still know that the point must be unstable. If that were the case we would require $b > \frac{5}{4}$ and $(b - \frac{5}{4})^2 \geq 1$, so that $b \geq \frac{9}{4}$.

c. [3 points] We said that the Brusselator can have oscillatory concentrations of x and y. Explain how the result here does (or does not) demonstrate this behavior.

Solution: We see that the critical point is unstable, and trajectories converge to something that looks like the limit cycle that we explored in the van der Pol equation in lab 2. On that limit cycle, the values of x and y will oscillate as we move around the closed trajectory.