Math 216 — First Midterm

18 October, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

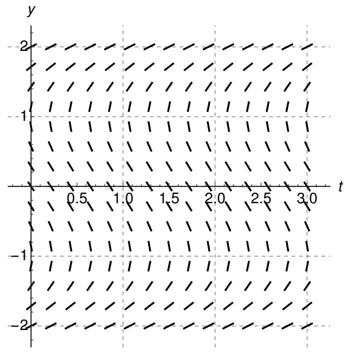
1. [12 points] Suppose that a bucket with a capacity of 20 liters containing 1.3 kg of sand (which has a volume of 1 liter) is left outside in a very heavy rainstorm with a rainfall rate of 10 cm/hour. For a standard bucket, this results in water being added to the bucket at a rate of about 7 liters/hour.[1]

a. [6 points] Until the bucket fills, the amount of sand in the bucket is constant. Suppose that the rain fills the bucket before the end of the storm. Write an initial value problem for the amount of sand in the bucket after the bucket fills. You may take $t = 0$ as the time at which the bucket fills, and should assume that the sand is uniformly distributed through the water in the bucket.

b. [6 points] What equilibrium solution, or solutions, does your equation in (a) have? Are they stable? Explain why this makes sense physically.

[1] For those who prefer English units, this is, approximately, a 5 gallon bucket with a bit less than 3 lb, or a quarter gallon, of sand. The rainfall is about 4 in/hour.

2. [15 points] Consider the direction field shown to the right, which corresponds to a first order differential equation $y' = f(t, y)$.



a. [5 points] Which of the following functions $f(t, y)$ is most likely to be the function in this differential equation? Briefly explain how you made your choice.

$f(t,y) = (y+1)(y-1)$ $\qquad$ $f(t,y) = \sin(\frac{\pi}{2}\,y)$

$f(t,y) = \frac{2}{(y+1)(y-1)}$ $\qquad$ $f(t,y) = \frac{2}{\sin(\frac{\pi}{2}\,y)}$

$f(t,y) = \frac{\sin(\frac{\pi}{2}\,t)}{y^2-1}$ $\qquad$ $f(t,y) = \frac{y+1}{y-1}$

b. [5 points] Sketch, on the direction field or below, the solution to $y' = f(t, y)$, $y(1) = 0$. For what values of t and y will it exist (you should be able to determine these without calculations)? Why?

c. [5 points] Based on your choice of $f(t, y)$ in (a) and the corresponding direction field, are there any initial conditions (t_0, y_0) for which you cannot guarantee that there exists a unique solution? Explain.

3. [15 points] Consider the equation $y' = ay - y^4$.

a. [5 points] Find all critical points for this equation.

b. [6 points] Draw a phase line for each of the cases $a = -2, 0, 2$. Determine the stability of the critical points in each case.

c. [4 points] Sketch a *bifurcation diagram* that shows the position of the critical points as a function of a in the ay-plane.

4. [12 points] For each of the following give an example, as indicated. It may be useful to note that the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ are $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda = 2$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a. [3 points] Give an example of a linear first-order equation that is not separable.

b. [3 points] Give two distinct, non-zero solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to the system $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$ for which $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is not a general solution to the system.

c. [3 points] Give a 2×2 matrix $\mathbf{B}$, with all non-zero entries, for which $\mathbf{B}\mathbf{x} = \mathbf{0}$ has an infinite number of solutions.

d. [3 points] Give three different vectors, $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, for which $\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{w}_j = k\,\mathbf{w}_j$, for some k. (The value of k need not be the same for all three vectors.)

5. [16 points] Recall that the van der Pol equation we studied in lab 1 is given by

$$x'' + \mu f'(x)x' + x = 0,$$

or, as a system in x and $y = x'$,

$$x' = y, \qquad y' = -x - \mu f'(x)y,$$

for some function $f'(x)$. We assume that $\mu > 0$.

a. [3 points] Show that for any choice of $f'(x)$, the only critical point of the system formulation of the van der Pol equation is $(0, 0)$.

b. [4 points] Suppose that the Taylor expansion of $f'(x)$ around $x = 0$ is $f'(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$. Use this expansion to linearize your system. Your linear system will involve the coefficients a_n.

Problem 5, continued. We are considering the van der Pol system.

c. [5 points] Suppose that the system you obtained in (b) is $x' = -a_0\mu x - y$, $y' = x$. For what value or values of a_0 will the phase portrait for this system have only one straight-line of solution trajectories? No straight-line trajectories? Explain.

d. [4 points] When a_0 is picked so that there is a single straight-line of solution trajectories in the phase portrait for this system, give an initial condition that will result in a straight-line trajectory in the phase plane.

6. [15 points] Find real-valued solutions to each of the following, as indicated. Where possible, find explicit solutions.

a. [7 points] Find the general solution to the Gompertz equation, $\frac{dy}{dt} = r\,y\,\ln(\frac{K}{y})$.

b. [8 points] Solve $y' = 3t - \frac{t}{1+t^2}\,y$, with $y(0) = 3$.

7. [15 points] Find explicit, real-valued solutions to each of the following, as indicated.

a. [7 points] Find the general solution to the system $x' = y$, $y' = 2x + y$.

b. [8 points] Solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.