Math 216 — Second Midterm

15 Nov, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [12 points] Find each of the following.

a. [7 points] Use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} e^{-1}, & 0 < t \leq 1 \\ e^{-t}, & 1 < t < \infty. \end{cases}$$

b. [5 points] Give another function $g(t)$ for which $\mathcal{L}\{g(t)\} = F(s)$. Explain your answer briefly.

2. [15 points] In each of the following $\mathcal{L}$ is the Laplace transform operator, and, in (b), L is a linear, constant-coefficient differential operator.

a. [5 points] If $x' = 3x + 4y$ and $y' = 2x - y$, with initial conditions $x(0) = 0$ and $y(0) = 2$, find $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

b. [5 points] Suppose that when solving an equation $L[y] = f(t)$, $y(0) = y_0$, $y'(0) = v_0$ using the Laplace transform, we find

$$\mathcal{L}\{y(t)\} = Y(s) = \frac{5}{(s+1)(s+2)} + \frac{s}{(s+1)(s+2)(s^2+4)}.$$

What are L, $f(t)$, and the initial conditions y_0 and v_0?

c. [5 points] Derive the transform rule $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ for a continuous function $f(t)$.

3. [16 points] For $t > 0$, consider the differential equation $L[y] = y'' - 3t^{-1}y' - 5t^{-2}y = 0$.

a. [4 points] Determine which of $y_1 = t^{-1}$, $y_2 = 1$, $y_3 = t$, $y_4 = \frac{1+t^6}{t}$, and $y_5 = t^5$ are solutions to $L[y] = 0$.

b. [4 points] Write a general solution to $L[y] = 0$. Explain why your solution is correct.

c. [4 points] If you were solving $L[y] = 5t^5$, what forms could the particular solution take (that is, what could you guess for y_p)? Why?

d. [4 points] Find y_p.

4. [15 points] Consider the homogeneous problem $L[y] = my'' + \gamma y' + ky = 0$.

a. [5 points] If this models critically damped harmonic motion, find the general solution to the problem.

b. [5 points] Sketch a phase portrait for the system for the case when this represents critically damped harmonic motion.

c. [5 points] Suppose that we decrease γ in our equation very slightly from the critically damped case we considered in (a) and (b). Sketch the phase portrait for the new system. Why does it change as it does? What type of damping are we seeing now?

5. [12 points] In lab 4 we consider a forced electrical system of the form

$$y'' + 2\gamma y' + \omega_0^2 y = F(t),$$

which models the current in a circuit. In this problem we take $\gamma = 1$ and $\omega_0 = 3$.

a. [7 points] Carefully sketch a qualitatively accurate graph of the steady state response current to a forcing voltage $F = k\sin(\omega t)$. Explain what functions appear in the response and therefore why your graph has the form it does. As possible, give information about the relative position of significant features of your graph. *(Note that you do not need to, and probably do not want to, solve for the steady state response.)*

b. [5 points] Now suppose that

$$F(t) = I(t) = \begin{cases} \frac{1}{a}, & c \le t < c + a \\ 0, & \text{otherwise,} \end{cases}$$

and that $y(0) = y'(0) = 0$. Make two sketches showing the behavior of the solution for $t > c$, first if a is large and second if a is small. In either case you will want to say something about what functions contribute to the behavior you are graphing, but need not, and probably should not, completely solve the problem.

6. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem do **not** use Laplace transforms. Note that you do not need to simplify numeric expressions.

a. [8 points] Find the solution to $3y'' + 4y' + y = 5e^{-t}$, $y(0) = y'(0) = 0$

b. [7 points] Find the general solution to $y'' + 2y' + 10y = 5t$.

7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem, **do** use Laplace transforms.

a. [8 points] Solve $y'' + 2y' + 10y = 5$, $y(0) = 1$, $y'(0) = 2$.

b. [7 points] Find the solution to $y'' + 5y' + 6y = e^{-3t}$, $y(0) = y'(0) = 0$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:

  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.

- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.

- Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$