Math 216 — Final Exam

14 Dec, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Suppose we are solving the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 0 \\ 8 \end{pmatrix}$.

a. [4 points] If $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$, find all critical points for the system.

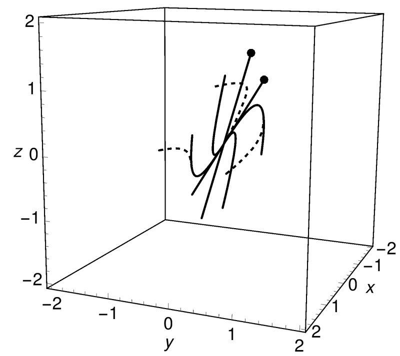
b. [5 points] If the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_{1,2} = -4, -2$ with $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, sketch a phase portrait for the system.

c. [3 points] For a different $\mathbf{A}$, could a solution to the system be $x = e^{-3t}\sin(t)$, $y = e^{-3t}\cos(t)$? Explain.

2. [10 points] In this problem we consider a linearization of the Lorenz system that we considered in lab 5, with $\eta = \sqrt{8(r-1)/3}$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\eta \\ \eta & \eta & -2.67 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For some value of r, if phase portrait is shown in the figure to the right, below, what can you say about the eigenvalues and eigenvectors of this system (please, whatever you do, do not try to calculate exact values from the system)? The solid black trajectories lie in a plane. The dashed trajectories start to the left the plane as you look at it. The two black points are, approximately, $(1, 0.75, 1.5)$ and $(1, 1, 1.25)$. You should be able to specify at least two eigenvectors and the relative values of the eigenvalues.



b. [5 points] For a different value of r, the general solution to the system is, approximately,

$$\begin{aligned} x &= c_1 e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ y &= -0.4\,c_1\,e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ z &= -0.1\,c_1\,e^{-12t} + 0.75\,c_2\,e^{-t}\cos(0.4\,t) + 0.75\,c_3\,e^{-t}\sin(0.4\,t). \end{aligned}$$

What are the eigenvalues and eigenvectors of the coefficient matrix for the linear system in this case?

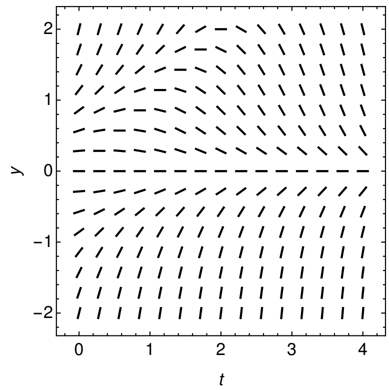
3. [12 points] In each of the following we consider a first order differential equation $y' = f(t, y)$. In these, the functions $f(t, y)$ and $g(t, y)$ are different functions.

a. [6 points] The direction field for the equation $y' = f(t, y)$ is shown to the right. For each of the following, explain if the statement is true, false, or if you cannot tell.

(1) The equation is autonomous, that is, $f(t, y)$ is actually only a function of y.

(2) The equation is linear.

(3) The initial value problem $y' = f(t, y)$, $y(0) = y_0$ has a unique solution for all y_0 between -2 and 2.



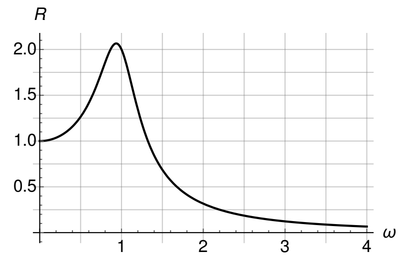
b. [6 points] Let $y' = g(t, y) = y(y^3 - a^3)$, where a is a real number. Identify all a for which it is true both there is a critical point other than $y = 0$, and that $y = 0$ is stable. Be sure it is clear how you arrive at your conclusion. Draw a phase line for this situation, or explain why it is impossible.

4. [12 points] In lab 4 we consider a forced electrical system of the form

$$y'' + 2\gamma y' + \omega_0^2 y = F(t),$$

which models the current in a circuit. In this problem we take $\gamma = 0.25$ and $\omega_0 = 1$.

a. [6 points] The amplitude of the long-term response to a forcing voltage $F = \sin(\omega t)$ is shown as a function of ω in the figure to the right. Without solving the differential equation, sketch a reasonably accurate graph of the solution to the differential equation with initial conditions $y(0) = 1$, $y'(0) = 0$, when $\omega = 0.5$. (Your graph should include magnitudes and timescales where it is possible to determine them.)



b. [6 points] Now suppose that

$$F(t) = I(t) = \begin{cases} 0, & t < c \\ k, & c \le t, \end{cases}$$

and that $y(0) = y'(0) = 0$. Without solving the differential equation, sketch the behavior of the solution, indicating the magnitude of the solution and its other characteristics as possible.

5. [12 points] Consider the systems model of a linear oscillator given by

$$x' = y, \quad y' = -2x - 2y + k\delta(t - t_0),$$

with initial conditions $x(0) = 0$, $y(0) = 2$.

a. [5 points] **Use Laplace transforms on the system** to find $x(t)$.

b. [5 points] For what t_0 and k will $x(t)$ be identically zero for all $t > t_0$?

c. [2 points] Give a physical system that this could model and explain what the result in (b) corresponds to in the model.

6. [12 points] Consider the nonlinear system

$$x' = 1 - y, \qquad y' = 2 - 2y + 3\sin(x).$$

Sketch a qualitatively accurate phase portrait showing representative trajectories, by doing appropriate linearization and local analysis. Use your phase portrait to predict the behavior of a trajectory starting at $x(0) = \pi$, $y(0) = 0$.

You SHOULD NOT complete this page or the one following it if you have completed the mastery assessment.

7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem **do NOT use Laplace transforms**. Note that you do not need to simplify numeric expressions.

a. [8 points] Find the solution to $y' = \dfrac{8 + 4s^2 - 2sy}{2 + s^2}$, $y(0) = 4$.

b. [7 points] Find the general solution to $y'' - 4y' - 5y = -10t + 22$.

You SHOULD NOT complete this page or the one preceding it if you have completed the mastery assessment.

8. [15 points] For each of the following find explicit, real-valued solutions as indicated.

a. [8 points] Solve $\mathbf{x}' = \begin{pmatrix} -4 & -10 \\ 10 & 8 \end{pmatrix} \mathbf{x}$.

b. [7 points] Find the solution to $z'' + 4z' + 20z = 8u_6(t)$, $z(0) = 0$, $z'(0) = 9$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\frac{s}{s^2 + a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t - c)$	e^{-cs}
A.	$f'(t)$	$s\, F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\, f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s - c)$
D.	$u_c(t)\, f(t - c)$	$e^{-cs}\, F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$
F.	$\int_0^t f(x) g(t - x)\, dx$	$F(s)G(s)$