Math 216 — First Midterm

18 October, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Suppose that a bucket with a capacity of 20 liters containing 1.3 kg of sand (which has a volume of 1 liter) is left outside in a very heavy rainstorm with a rainfall rate of 10 cm/hour. For a standard bucket, this results in water being added to the bucket at a rate of about 7 liters/hour.[1]

a. [6 points] Until the bucket fills, the amount of sand in the bucket is constant. Suppose that the rain fills the bucket before the end of the storm. Write an initial value problem for the amount of sand in the bucket after the bucket fills. You may take $t = 0$ as the time at which the bucket fills, and should assume that the sand is uniformly distributed through the water in the bucket.

Solution: Let x be the amount of sand in the bucket, in kg. Then $x(0) = 1.3$. No sand is added to the bucket, but it is lost as the bucket overflows. Water will leave the bucket at the same rate as the rain is entering the bucket, 7 liters/hr. There are 20 liters in the bucket, so the concentration of sand in the bucket is $\frac{x}{20}$ kg/liter, and the differential equation for x is

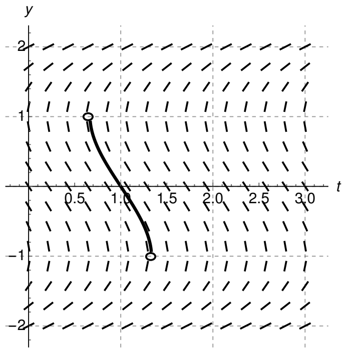
$$\frac{dx}{dt} = -7 \cdot \frac{x}{20} = -\frac{7}{20}\,x.$$

b. [6 points] What equilibrium solution, or solutions, does your equation in (a) have? Are they stable? Explain why this makes sense physically.

Solution: Equilibrium solutions are given by $\frac{dx}{dt} = f(x) = -\frac{7}{20}\,x = 0$, which is true if $x = 0$. Note that $f'(x) = -\frac{7}{20} < 0$, so this is a stable equilibrium. This makes sense: if it keeps raining forever, the sand will slowly get washed out of the bucket, leaving nothing but water.

[1] For those who prefer English units, this is, approximately, a 5 gallon bucket with a bit less than 3 lb, or a quarter gallon, of sand. The rainfall is about 4 in/hour.

2. [15 points] Consider the direction field shown to the right, which corresponds to a first order differential equation $y' = f(t, y)$.



a. [5 points] Which of the following functions $f(t, y)$ is most likely to be the function in this differential equation? Briefly explain how you made your choice.

$f(t,y) = (y+1)(y-1)$	$f(t,y) = \sin(\frac{\pi}{2}\, y)$
$\boxed{f(t,y) = \frac{2}{(y+1)(y-1)}}$	$f(t,y) = \frac{2}{\sin(\frac{\pi}{2}\, y)}$
$f(t,y) = \frac{\sin(\frac{\pi}{2}\, t)}{y^2-1}$	$f(t,y) = \frac{y+1}{y-1}$

Solution: We note that the direction field shows vertical slopes at $y = \pm 1$, which suggests that $f(t, y)$ is undefined there; this indicates that the answer is one of $f(t,y) = \frac{2}{(y+1)(y-1)}$, or $f(t,y) = \frac{\sin(\frac{\pi}{2}\, t)}{y^2-1}$. Further, the direction field is unchanged under translations in time, so it cannot be the latter. Therefore, the answer must be $f(t,y) = \frac{2}{(y+1)(y-1)}$.

b. [5 points] Sketch, on the direction field or below, the solution to $y' = f(t, y)$, $y(1) = 0$. For what values of t and y will it exist (you should be able to determine these without calculations)? Why?

Solution: We can trace a solution by using the fact that at every point in the (t, y) plane the solution must be parallel to the direction field ticks. This gives the result shown above. We note that at $y = \pm 1$ the solution curve reaches the point where its slope becomes undefined, and after that to follow the direction field we would end up with a curve that is not a function. Thus we expect that the solution will exist only for $-1 < y < 1$, which, from our sketch, is approximately $0.67 < t < 1.33$.

c. [5 points] Based on your choice of $f(t, y)$ in (a) and the corresponding direction field, are there any initial conditions (t_0, y_0) for which you cannot guarantee that there exists a unique solution? Explain.

Solution: We know that there will be a unique solution through any initial condition (t_0, y_0) where the function f and its partial derivative with respect to y, f_y, are continuous. From (a), this is whenever $y \neq \pm 1$, which corresponds in the direction field to where the slopes are defined (not vertical). Thus, for any initial condition (t_0, y_0) with $y_0 \neq \pm 1$ we will have a unique solution, but it may exist on a limited interval of time depending on whether or when it reaches one of the horizontal lines $y = \pm 1$. If we start with $y(t_0) = \pm 1$, the differential equation is not well defined, so it is not clear what we should do with it. If we assume that we can deal with that (e.g., by clearing the denominator), if $y_0 = \pm 1$ we might expect a non-unique solution.

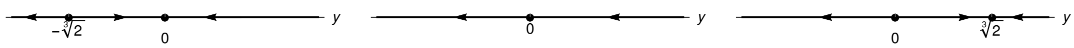
3. [15 points] Consider the equation $y' = ay - y^4$.

a. [5 points] Find all critical points for this equation.

Solution: Critical points are where $y' = 0 = y(a - y^3)$, so that $y = 0$ or $y = a^{1/3}$.

b. [6 points] Draw a phase line for each of the cases $a = -2, 0, 2$. Determine the stability of the critical points in each case.

Solution: If $a < 0$, $y' = -|a|y - y^4$, so that $y' < 0$ if $y < -|a^{1/3}|$, $y' > 0$ if $-|a^{1/3}| < y < 0$, and $y' < 0$ if $y > 0$. If $a = 0$, there is only one critical point, $y = 0$, and $y' = -y^4 < 0$ for all $y \neq 0$. If $a > 0$, $y' = ay - y^4 < 0$ if $y < 0$, $y' > 0$ if $0 < y < a^{1/3}$, and $y' < 0$ if $y > a^{1/3}$. This gives the three phase lines shown below.

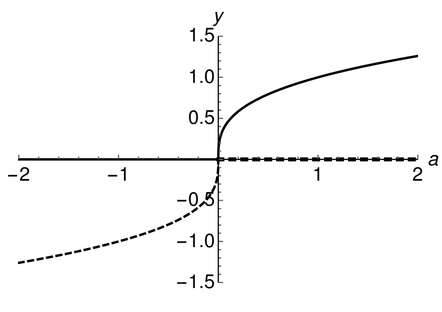


$a = -2$ $a = 0$ $a = 2$

Thus, if $a < 0$, the critical point $y = a^{1/3}$ is unstable and $y = 0$ is asymptotically stable. If $a = 0$, $y = 0$ is unstable (or semi-stable). If $a > 0$, $y = a^{1/3}$ is asymptotically stable and $y = 0$ is unstable.

c. [4 points] Sketch a *bifurcation diagram* that shows the position of the critical points as a function of a in the ay-plane.

Solution: The bifurcation diagram is shown below. We indicate stable critical points with thick black curves (these are $y = 0$ for $a < 0$, and $y = a^{1/3}$ for $a > 0$, and unstable ones with dashed curves (these are $y = -|a|^{1/3}$ for $a < 0$ and $y = 0$ for $a > 0$).



4. [12 points] For each of the following give an example, as indicated. It may be useful to note that the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ are $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\lambda = 2$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a. [3 points] Give an example of a linear first-order equation that is not separable.

Solution: There are many possible answers; one is

$$y' + ty = \sin(t).$$

Note that any autonomous equation is necessarily separable.

b. [3 points] Give two distinct, non-zero solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ to the system $\mathbf{x}' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$ for which $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is not a general solution to the system.

Solution: This just requires that we have two linearly dependent solutions. Using the eigenvalues above, one solution is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$ and a second, linearly dependent, solution is $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{2t}$.

c. [3 points] Give a 2×2 matrix $\mathbf{B}$, with all non-zero entries, for which $\mathbf{Bx} = \mathbf{0}$ has an infinite number of solutions.

Solution: Note that $\mathbf{x} = \mathbf{0}$ is always a solution, so we need only for $\mathbf{B}$ to be singular—that is, $\det(\mathbf{B}) = 0$. If $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this requires that $ac - bd = 0$. One such matrix is $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$.

d. [3 points] Give three different vectors, $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, for which $\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{w}_j = k\,\mathbf{w}_j$, for some k. (The value of k need not be the same for all three vectors.)

Solution: This is just asking for an eigenvector-eigenvalue pair for the matrix. From above, we may use $k = 1$ and $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $k = 2$ and $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $k = 1$ or $k = 2$ and some multiple of the corresponding eigenvector, for example, $k = 1$ and $\mathbf{w}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. We could also take $k = 1$ and three different multiples of $\mathbf{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$.

5. [16 points] Recall that the van der Pol equation we studied in lab 1 is given by

$$x'' + \mu f'(x)x' + x = 0,$$

or, as a system in x and $y = x'$,

$$x' = y, \qquad y' = -x - \mu f'(x)y,$$

for some function $f'(x)$. We assume that $\mu > 0$.

a. [3 points] Show that for any choice of $f'(x)$, the only critical point of the system formulation of the van der Pol equation is $(0, 0)$.

Solution: At a critical point, $x' = y' = 0$. Thus, from the first equation, $y = 0$. Then, from the second, $y' = 0 = -x - \mu f'(x) \cdot 0$, and $x = 0$ as well.

b. [4 points] Suppose that the Taylor expansion of $f'(x)$ around $x = 0$ is $f'(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots$. Use this expansion to linearize your system. Your linear system will involve the coefficients a_n.

Solution: Note that the first equation is already linear. The second is $y' = -x - \mu(a_0 + a_1 x + \cdots)y$, so that, dropping nonlinear terms, the linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -\mu a_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Problem 5, continued. We are considering the van der Pol system.

c. [5 points] Suppose that the system you obtained in (b) is $x' = -a_0\mu x - y$, $y' = x$. For what value or values of a_0 will the phase portrait for this system have only one straight-line of solution trajectories? No straight-line trajectories? Explain.

Solution: There is a single straight-line solution if the coefficient matrix has a repeated eigenvalue, and no straight-line solutions if the eigenvalues are complex. Here, eigenvalues are given by

$$\det(\begin{pmatrix} -\mu a_0 - \lambda & -1 \\ 1 & -\lambda \end{pmatrix}) = \lambda^2 + (\mu a_0)\lambda + 1 = (\lambda + \frac{1}{2}\mu a_0)^2 + 1 - \frac{1}{4}\mu^2 a_0^2 = 0.$$

This gives $\lambda = -\frac{1}{2}\mu a_0 \pm \frac{1}{2}\sqrt{\frac{1}{4}\mu^2 a_0^2 - 1}$. Thus, there will be a single straight-line solution if $a_0 = \pm\frac{2}{\mu}$, and no straight-line solutions if $|a_0| < \frac{2}{\mu}$, so that $\frac{1}{4}\mu a_0^2 - 1 < 0$.

d. [4 points] When a_0 is picked so that there is a single straight-line of solution trajectories in the phase portrait for this system, give an initial condition that will result in a straight-line trajectory in the phase plane.

Solution: This is when $a_0 = \pm\frac{2}{\mu}$, so that $\lambda = -\frac{1}{2}\mu a_0 = \mp 1$. If $a_0 = \frac{2}{\mu}$, the eigenvector satisfies the equation $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus any initial condition on the line $y = -x$ will converge along that line to the origin.

6. [15 points] Find real-valued solutions to each of the following, as indicated. Where possible, find explicit solutions.

a. [7 points] Find the general solution to the Gompertz equation, $\frac{dy}{dt} = r\,y\,\ln(\frac{K}{y})$.

Solution: We rewrite this using rules of logarithms as $\frac{dy}{dt} = ry(\ln(K) - \ln(y))$, and separate to have

$$\frac{dy/dt}{y(\ln(y) - \ln(K))} = -r.$$

Integrating both sides, we have

$$\ln|\ln(y) - \ln(K)| = -rt + \hat{C},$$

so that $\ln(y) - \ln(K) = Ce^{-rt}$ (with $C = \pm e^{\hat{C}}$). Adding $\ln(K)$ to both sides and exponentiating again,

$$y = Ke^{C\exp(-rt)}.$$

b. [8 points] Solve $y' = 3t - \frac{t}{1+t^2}\,y$, with $y(0) = 3$.

Solution: This is linear and not separable. We rewrite it in standard form to get $y' + \frac{t}{1+t^2}y = 3t$, and an integrating factor is $\mu = \exp(\int t/(1+t^2)\,dt) = \exp(\ln(1+t^2)/2 = \sqrt{1+t^2}$. Multiplying by μ, we have

$$\left(y\,\sqrt{1+t^2}\right)' = 3t\sqrt{1+t^2},$$

so that on integrating both sides, $y\,\sqrt{1+t^2} = (1+t^2)^{3/2} + C$, and

$$y = 1 + t^2 + \frac{C}{\sqrt{1+t^2}}.$$

With $y(0) = 3$, we have $y(0) = 3 = 1 + C$, and $C = 2$, so

$$y = 1 + t^2 + \frac{2}{\sqrt{1+t^2}}.$$

7. [15 points] Find explicit, real-valued solutions to each of the following, as indicated.

a. [7 points] Find the general solution to the system $x' = y$, $y' = 2x + y$.

Solution: Written as a matrix system, the coefficient matrix on the right-hand side is $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$, so that eigenvalues satisfy

$$\det(\begin{pmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix}) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0.$$

Thus $\lambda = -1$ or $\lambda = 2$. If $\lambda = -1$, the eigenvector satisfies $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ (or any nonzero constant multiple of this). If $\lambda = 2$, we have $\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$ (or any constant multiple thereof). Thus the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ -c_1 e^{-t} + 2c_2 e^{2t} \end{pmatrix}.$$

b. [8 points] Solve $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Solution: Eigenvalues of the coefficient matrix are given by

$$\det(\begin{pmatrix} -\lambda & 1 \\ -2 & 2-\lambda \end{pmatrix}) = \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0.$$

Thus $\lambda = 1 \pm i$. We consider $\lambda = 1 + i$, and use it to obtain two linearly independent real-valued solutions. In this case, the eigenvector satisfies $\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that one choice is $\mathbf{v} = \begin{pmatrix} 1 & 1+i \end{pmatrix}^T$. (Using the second equation gives $\mathbf{v} = \begin{pmatrix} 1-i & 2 \end{pmatrix}^T$.) We write the corresponding complex-valued solution and obtain real-valued solutions by taking the real and imaginary parts:

$$\mathbf{x} = \begin{pmatrix} e^t(\cos(t) + i\sin(t)) \\ e^t(\cos(t) + i\sin(t) + i\cos(t) - \sin(t)) \end{pmatrix},$$

so that a general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t.$$

We note that if $c_1 = 0$, $c_2 = 2$ the initial condition is satisfied; thus $\mathbf{x} = 2 \begin{pmatrix} \sin(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t$. With the second $\mathbf{v}$, above, the complex-valued form of and general solution for $\mathbf{x}$ are $\mathbf{x} = \begin{pmatrix} e^t(\cos(t) + i\sin(t) - i\cos(t) + \sin(t)) \\ e^t(2\cos(t) + 2i\sin(t)) \end{pmatrix}$ and $\mathbf{x} = c_1 \begin{pmatrix} \cos(t) + \sin(t) \\ 2\cos(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} -\cos(t) + \sin(t) \\ 2\sin(t) \end{pmatrix} e^t$, respectively. In this case $c_1 = c_2 = 1$ to give the solution above.