Math 216 — Second Midterm

15 Nov, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [12 points] Find each of the following.

a. [7 points] Use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} e^{-1}, & 0 < t \leq 1 \\ e^{-t}, & 1 < t < \infty. \end{cases}$$

Solution: Applying the transform, we have

$$\begin{aligned} F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^1 e^{-1}e^{-st}\,dt + \int_1^\infty e^{-(s+1)t}\,dt \\ &= -\frac{1}{s}\,e^{-1}e^{-st}\Big|_{t=0}^{t=1} - \frac{1}{s+1}\,e^{-(s+1)t}\Big|_{t=1}^{t\to\infty} \\ &= \frac{1}{s}(e^{-1} - e^{-(s+1)}) + \frac{1}{s+1}\,e^{-(s+1)}. \end{aligned}$$

b. [5 points] Give another function $g(t)$ for which $\mathcal{L}\{g(t)\} = F(s)$. Explain your answer briefly.

Solution: Because the transform is defined as an integral, any single point differences between g and f will not change the transform. Thus, we could take

$$g(t) = \begin{cases} e^{-1}, & 0 < t < 1 \\ 1, & t = 1 \\ e^{-t}, & 1 < t < \infty, \end{cases}$$

or any other variation where we make $g(t)$ piecewise continuous with isolated points that have a different value from $f(t)$.

2. [15 points] In each of the following $\mathcal{L}$ is the Laplace transform operator, and, in (b), L is a linear, constant-coefficient differential operator.

a. [5 points] If $x' = 3x + 4y$ and $y' = 2x - y$, with initial conditions $x(0) = 0$ and $y(0) = 2$, find $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

Solution: Taking the forward transform of both equations, we have $sX - 0 = 3X + 4Y$, $sY - 2 = 2X - Y$. Rewriting as a linear algebraic system, we have

$$\begin{aligned}(s-3)X - 4Y &= 0\\ -2X + (s+1)Y &= 2.\end{aligned}$$

Taking $(s+1)$ times the first and 4 times the second and adding, we find

$$X = \frac{8}{(s+1)(s-3)-8} = \frac{8}{s^2-2s-11}.$$

Similarly, taking 2 times the first and $(s-3)$ times the second and adding, we get

$$Y = \frac{2(s-3)}{(s-3)(s+1)-8} = \frac{2(s-3)}{s^2-2s-11}.$$

b. [5 points] Suppose that when solving an equation $L[y] = f(t)$, $y(0) = y_0$, $y'(0) = v_0$ using the Laplace transform, we find

$$\mathcal{L}\{y(t)\} = Y(s) = \frac{5}{(s+1)(s+2)} + \frac{s}{(s+1)(s+2)(s^2+4)}.$$

What are L, $f(t)$, and the initial conditions y_0 and v_0?

Solution: The factors $(s+1)(s+2) = s^2 + 3s + 2$ in both terms of the denominator indicate that $L = D^2 + 3D + 2$. The second term is the response to the forcing term $f(t)$, for which we see $\mathcal{L}\{f(t)\} = \frac{s}{s^2+4}$, so $f(t) = \cos(2t)$. Finally, the first term on the right-hand side is the response to the initial forcing. We see that there is no s in the numerator, so $y_0 = 0$; then $v_0 = 5$ to give the desired form.

c. [5 points] Derive the transform rule $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ for a continuous function $f(t)$.

Solution: We apply the integral definition of the transform:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt.$$

Integrating by parts with $u = e^{-st}$ and $v' = f'(t)$, we have

$$\int_0^\infty f'(t)\, e^{-st}\, dt = e^{-st} f(t)\Big|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} f(t)\, dt = -f(0) + s\mathcal{L}\{f(t)\},$$

assuming that the limit in the first term is well behaved as $t \to \infty$.

3. [16 points] For $t > 0$, consider the differential equation $L[y] = y'' - 3t^{-1}y' - 5t^{-2}y = 0$.

a. [4 points] Determine which of $y_1 = t^{-1}$, $y_2 = 1$, $y_3 = t$, $y_4 = \frac{1+t^6}{t}$, and $y_5 = t^5$ are solutions to $L[y] = 0$.

Solution: Note that $y_4 = t^{-1} + t^5 = y_1 + y_5$, so, because the operator is linear, y_4 will be a solution if y_1 and y_5 are. Plugging in, we see that

$$\begin{aligned} L[y_1] &= 2t^{-3} + 3t^{-3} - 5t^{-3} = 0, & L[y_2] &= 0 - 0 - 5t^{-2} \neq 0 \\ L[y_3] &= 0 - 3t^{-1} - 5t^{-1} \neq 0, \quad \text{and} & L[y_5] &= 20t^3 - 15t^3 - 5t^3 = 0. \end{aligned}$$

Thus y_1, y_4, and y_5 are solutions.

b. [4 points] Write a general solution to $L[y] = 0$. Explain why your solution is correct.

Solution: We need two linearly independent solutions to write a general solution, and $W[y_1, y_5] = \begin{vmatrix} t^{-1} & t^5 \\ -t^{-2} & 5t^4 \end{vmatrix} = 6t^3 \neq 0$ (for $t > 0$), so they are linearly independent. A general solution is

$$y = c_1 t^{-1} + c_2 t^5.$$

c. [4 points] If you were solving $L[y] = 5t^5$, what forms could the particular solution take (that is, what could you guess for y_p)? Why?

Solution: This is a non-constant coefficient problem, so we do not expect that the method of undetermined coefficients will work. (With some effort, we could come up with a polynomial guess that will work, but this doesn't follow the rules we have for the method.) Therefore, it makes best sense to use variation of parameters. The form of the solution is then $y_p = u_1(t)t^{-1} + u_2(t)t^5$.

d. [4 points] Find y_p.

Solution: We know that $t^{-1}u_1' + t^5u_2' = 0$, and $-t^{-2}u_1' + 5t^4u_2' = 5t^5$. The first gives $u_1' = -t^6u_2'$, so, plugging into the second, $6t^4u_2' = 5t^5$. This gives $u_2' = \frac{5}{6}t$, so $u_1' = -\frac{5}{6}t^7$. Integrating, $u_1 = -\frac{5}{48}t^8$ and $u_2 = \frac{5}{12}t^2$. Thus

$$y_p = -\frac{5}{48}\,t^8\,t^{-1} + \frac{5}{12}\,t^2\,t^5 = \frac{5}{16}\,t^7.$$

If we memorized the formula for variation of parameters, we have (recalling that $W[y_1, y_2] = 6t^3$)

$$\begin{aligned} y_p &= -t^{-1}\int \frac{(t^5)(5t^5)}{6t^3}\,dt + t^5\int \frac{(t^{-1})(5t^5)}{6t^3}\,dt \\ &= -t^{-1}\int \frac{5}{6}\,t^7\,dt + t^5\int \frac{5}{6}\,t\,dt = \frac{5}{16}\,t^7 \end{aligned}$$

(where we have used the fact that the two integrals are exactly those we just calculated above).

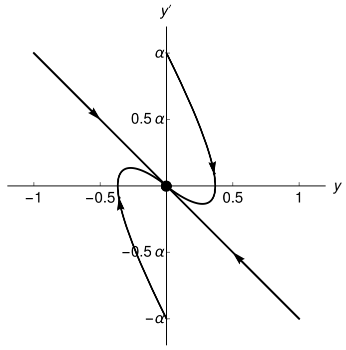
4. [15 points] Consider the homogeneous problem $L[y] = my'' + \gamma y' + ky = 0$.

a. [5 points] If this models critically damped harmonic motion, find the general solution to the problem.

Solution: If this is critically damped, there is a repeated real root to the characteristic equation $mr^2 + \gamma r + k = 0$. Thus, the characteristic equation is of the form $r^2 + 2\alpha r + \alpha^2$, so that $r = -\alpha$. The general solution is then given by $y = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$. Relating this to the original equation, $\alpha = \frac{\gamma}{2m}$, and in this case $\alpha^2 = \frac{k}{m}$.

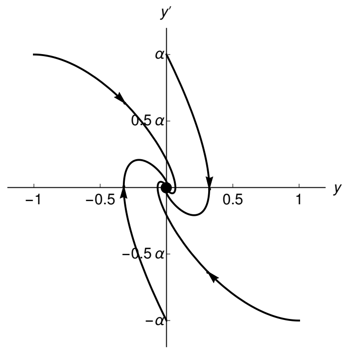
b. [5 points] Sketch a phase portrait for the system for the case when this represents critically damped harmonic motion.

Solution: In this case, the eigenvector for the system is $\mathbf{v} = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}$. Note that the second solution gives the vector form $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix} t e^{-\alpha t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\alpha t}$. Thus there is a single straight line in the phase portrait, $y = -\alpha x$, along which solutions approach the origin. If we consider an initial condition $y(0) = 0$, $y'(0) = 1$, the trajectory will be described by the second solution, and will initially increase in the direction $\mathbf{v}$ and then collapse to the origin. This gives the phase portrait shown below.



c. [5 points] Suppose that we decrease γ in our equation very slightly from the critically damped case we considered in (a) and (b). Sketch the phase portrait for the new system. Why does it change as it does? What type of damping are we seeing now?

Solution: If γ decreases slightly, then solutions of the characteristic equation will become complex valued, $r = -\frac{\gamma}{2m} \pm i\frac{1}{2m}\sqrt{4mk - \gamma^2} = -\mu \pm i\nu$, and the general solution will be $y = c_1 e^{-\mu t}\cos(\nu t) + c_2 e^{-\mu t}\sin(\nu t)$. The phase portrait will then be spirals, with no straight line solutions, becoming something like the figure below. The system is in this case underdamped.



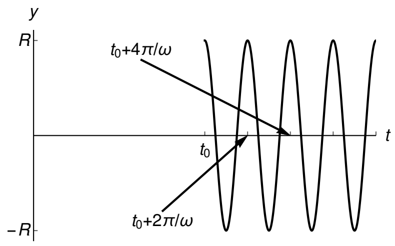
5. [12 points] In lab 4 we consider a forced electrical system of the form

$$y'' + 2\gamma y' + \omega_0^2 y = F(t),$$

which models the current in a circuit. In this problem we take $\gamma = 1$ and $\omega_0 = 3$.

a. [7 points] Carefully sketch a qualitatively accurate graph of the steady state response current to a forcing voltage $F = k\sin(\omega t)$. Explain what functions appear in the response and therefore why your graph has the form it does. As possible, give information about the relative position of significant features of your graph. *(Note that you do not need to, and probably do not want to, solve for the steady state response.)*

Solution: We know that the solution to the problem will be $y = y_c + y_p$, and because of the damping term $2\gamma y'$, the homogeneous solution will decay. Thus the steady state response will be y_p, which from the method of undetermined coefficients will be $y_p = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \phi)$. Thus the steady state response is a pure sinusoidal solution with period $T = \frac{2\pi}{\omega}$, and the amplitude $R = \sqrt{A^2 + B^2}$ will depend on ω as well. Thus we have a graph such as that below. Before some starting value t_0, we will have the sum of this steady solution and the (decaying) homogenous solution.

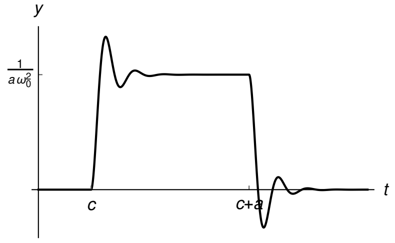


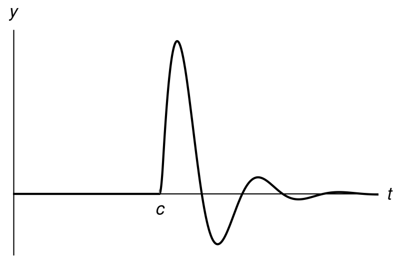
b. [5 points] Now suppose that

$$F(t) = I(t) = \begin{cases} \frac{1}{a}, & c \le t < c + a \\ 0, & \text{otherwise,} \end{cases}$$

and that $y(0) = y'(0) = 0$. Make two sketches showing the behavior of the solution for $t > c$, first if a is large and second if a is small. In either case you will want to say something about what functions contribute to the behavior you are graphing, but need not, and probably should not, completely solve the problem.

Solution: In either case, for $c < t < c + a$ we have $y_p = \frac{1}{a\omega_0^2}$, overlayed with the homogeneous term, which will be a decaying sinusoid. For $t > c + a$, the response will decay to zero, in an oscillatory manner. If we let $a \to 0$, we know from lab that this will result in a decaying sinusoid that is equivalent to the solution to the problem with zero forcing and initial condition $y(0) = 0$, $y'(0) = 1$. Thus we have the two graphs shown below.





6. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem do **not** use Laplace transforms. Note that you do not need to simplify numeric expressions.

a. [8 points] Find the solution to $3y'' + 4y' + y = 5e^{-t}$, $y(0) = y'(0) = 0$

Solution: We look for a homogeneous solution of the form $y = e^{rt}$, so that $3r^2+4r+1 = (3r+1)(r+1) = 0$, and $r = -\frac{1}{3}$ or $r = -1$. The complementary homogeneous solution is therefore $y = c_1e^{-t/3} + c_2e^{-t}$. For the particular solution we look for $y = Ate^{-t}$, because the exponential is part of the homogeneous solution. Plugging this into the equation, we have

$$3(-2Ae^{-t} + Ate^{-t} + 4(Ae^{-t} - Ate^{-t}) + Ate^{-t} = -2Ae^{-t} = 5e^{-t},$$

so that $A = -\frac{5}{2}$. Our general solution is therefore

$$y = c_1e^{-t/3} + c_2e^{-t} - \frac{5}{2}\,te^{-t}.$$

The initial condition requires that $c_1 + c_2 = 0$ and $-\frac{1}{3}c_1 - c_2 - \frac{5}{2} = 0$. Thus (adding), $c_1 = \frac{15}{4}$ and $c_2 = -\frac{15}{4}$, so that

$$y = \frac{15}{4}e^{-t/3} - \frac{15}{4}e^{-t} - \frac{5}{2}\,te^{-t}.$$

b. [7 points] Find the general solution to $y'' + 2y' + 10y = 5t$.

Solution: The homogeneous solution is given by $y = e^{rt}$, so that $r^2 + 2r + 10 = (r + 1)^2 + 9 = 0$, and $r = -1 \pm 3i$. The homogeneous solution is therefore $y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t)$. The particular solution is $y_p = At + B$, so that $A = \frac{1}{2}$ and $B = -\frac{1}{10}$, so that the general solution is

$$y = c_1e^{-t}\cos(3t) + c_2e^{-t}\sin(3t) + \frac{1}{2}t - \frac{1}{10}.$$

7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem, **do** use Laplace transforms.

a. [8 points] Solve $y'' + 2y' + 10y = 5$, $y(0) = 1$, $y'(0) = 2$.

Solution: With $Y = \mathcal{L}\{y\}$, the forward transform gives $s^2Y - 2 - s + 2sY - 2 + 10Y = \frac{5}{s}$, so that

$$Y = \frac{s+4}{(s+1)^2+9} + \frac{5}{s((s+1)^2+9)}.$$

To invert the second of these, we use partial fractions, letting $\frac{A}{s} + \frac{Bs+C}{(s+1)^2+9} = \frac{5}{s((s+1)^2+9)}$. Clearing the denominator, we must have $A(s^2 + 2s + 10) + Bs^2 + Cs = 5$. Thus $A = \frac{1}{2}$; matching terms in s^2, $B = -A = -\frac{1}{2}$; and matching terms in s, $C = -2A = -1$. We can therefore rewrite Y as

$$Y = \frac{(s+1)+3}{(s+1)^2+9} + \frac{1}{2s} - \frac{1}{2}\frac{(s+1)+1}{(s+1)^2+9}.$$

Inverting, we have

$$y = e^{-t}\cos(3t) + e^{-t}\sin(3t) + \frac{1}{2} - \frac{1}{2}e^{-t}\cos(3t) - \frac{1}{6}e^{-t}\sin(3t),$$

or, combining terms,

$$y = \frac{1}{2}e^{-t}\cos(3t) + \frac{5}{6}e^{-t}\sin(3t) + \frac{1}{2}.$$

b. [7 points] Find the solution to $y'' + 5y' + 6y = e^{-3t}$, $y(0) = y'(0) = 0$.

Solution: As before, the forward transform gives $(s^2 + 5s + 6)Y = \frac{1}{s+3}$, so that $Y = \frac{1}{(s+2)(s+3)^2}$. Using partial fractions,

$$\frac{1}{(s+2)(s+3)^2} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{(s+3)^2}.$$

Clearing denominators, $A(s+3)^2 + B(s+2)(s+3) + C(s+2) = 1$. Plugging in $s = -2$, $A = 1$; plugging in $s = -3$, $C = -1$. Finally, with $s = -1$, $4A + 2B + C = 3 + 2B = 1$, and $B = -1$. Thus

$$y = \mathcal{L}^{-1}\{\frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{(s+3)^2}\} = e^{-2t} - e^{-3t} - te^{-3t}.$$

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\frac{s}{s^2 + a^2}$
A.	$f'(t)$	$s\, F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\, f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$