Math 216 — Final Exam

14 Dec, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Suppose we are solving the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} 0 \\ 8 \end{pmatrix}$.

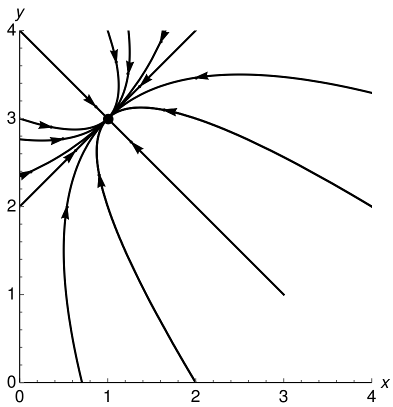
a. [4 points] If $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$, find all critical points for the system.

Solution: These are where $\mathbf{x}$ is a constant, so that, with $\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T$, $-3x + y = 0$ and $x - 3y = -8$. Thus $-8x = -8$, and $x = 1$, so that $y = 3$. The only critical point is $(1, 3)$.

b. [5 points] If the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_{1,2} = -4, -2$ with $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, sketch a phase portrait for the system.

Solution: From our work in (a) we know that the critical point for this system is $\mathbf{x}_c = (1, 3)$. Then, either by letting $\mathbf{x} = \mathbf{x}_c + \mathbf{u}$ (so that $\mathbf{u}' = \mathbf{A}\mathbf{u}$) or by noting that the general solution is $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = \mathbf{u} + \mathbf{x}_c$, we know that the phase portrait for the homogeneous system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ will be centered on $\mathbf{x}_c$.

The phase portrait there will have two straight line solutions that correspond to the eigenvectors, with solutions converging to $\mathbf{x}_c$. The eigenvector associated with $\mathbf{v}_1^T = \begin{pmatrix} -1 & 1 \end{pmatrix}$ is much smaller (more negative) than the other, so trajectories will collapse in this direction first, and then converge to the critical point along $\mathbf{v}_2^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$, giving the phase portrait below.



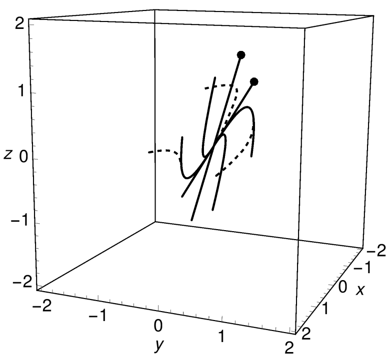
c. [3 points] For a different $\mathbf{A}$, could a solution to the system be $x = e^{-3t}\sin(t)$, $y = e^{-3t}\cos(t)$? Explain.

Solution: This is not possible, because of the shift in the critical point induced by the inhomogeneity. If we had $x = e^{-3t}\sin(t) + 1$, $y = e^{-3t}\cos(t) + 3$, following the logic given in the first paragraph in the solution for part (b), it could be the solution for a different matrix $\mathbf{A}$.

2. [10 points] In this problem we consider a linearization of the Lorenz system that we considered in lab 5, with $\eta = \sqrt{8(r-1)/3}$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & -\eta \\ \eta & \eta & -2.67 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

a. [5 points] For some value of r, if phase portrait is shown in the figure to the right, below, what can you say about the eigenvalues and eigenvectors of this system (please, whatever you do, do not try to calculate exact values from the system)? The solid black trajectories lie in a plane. The dashed trajectories start to the left the plane as you look at it. The two black points are, approximately, $(1, 0.75, 1.5)$ and $(1, 1, 1.25)$. You should be able to specify at least two eigenvectors and the relative values of the eigenvalues.



Solution: Note that all eigenvalues must be real, because there are two straight line trajectories (and thus there must be three). We assume that trajectories decay to the origin. Then, because the dashed trajectories decay rapidly into the plane determined by the solid trajectories, we know that one eigenvalue is much more negative than the other two. Given the two points, which appear to lie on straight-line solutions in the plane, we know that two eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 & 0.75 & 1.5 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 & 1.25 \end{pmatrix}^T$ (note that $\mathbf{v}_1$ has the larger z-coordinate), with eigenvalues $\lambda_1 < \lambda_2 < 0$. The third eigenvalue, λ_3, is significantly more negative than λ_1.

If we assume solutions grow instead of decaying we have the same analysis, but with $0 < \lambda_2 < \lambda_1 < \lambda_3$.

b. [5 points] For a different value of r, the general solution to the system is, approximately,

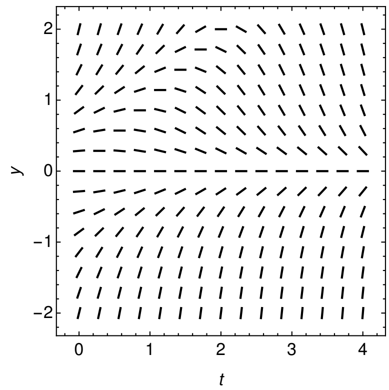
$$\begin{aligned} x &= c_1 e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ y &= -0.4\,c_1\,e^{-12t} + c_2\,(0.5\,\cos(0.4\,t) - 0.05\,\sin(0.4\,t))\,e^{-t} + \\ &\qquad c_3\,(0.05\,\cos(0.4\,t) + 0.5\,\sin(0.4\,t))\,e^{-t} \\ z &= -0.1\,c_1\,e^{-12t} + 0.75\,c_2\,e^{-t}\cos(0.4\,t) + 0.75\,c_3\,e^{-t}\sin(0.4\,t). \end{aligned}$$

What are the eigenvalues and eigenvectors of the coefficient matrix for the linear system in this case?

Solution: We see that $\lambda_1 = -12$ and $\lambda_{2,3} = -1 \pm 0.4i$. The eigenvector $\mathbf{v}_1$ we can read from the first component of the solutions: $\mathbf{v}_1 = \begin{pmatrix} 1 & -0.4 & -0.1 \end{pmatrix}^T$. For the complex λ, we can read the real and imaginary parts of each component of the vector from the coefficient of the cosine terms in the second and third solutions, getting $\mathbf{v}_{2,3} = \begin{pmatrix} 0.5 \pm 0.05i & 0.5 \pm 0.05i & 0.75 \end{pmatrix}^T$.

3. [12 points] In each of the following we consider a first order differential equation $y' = f(t, y)$. In these, the functions $f(t, y)$ and $g(t, y)$ are different functions.

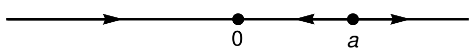
a. [6 points] The direction field for the equation $y' = f(t, y)$ is shown to the right. For each of the following, explain if the statement is true, false, or if you cannot tell.
(1) The equation is autonomous, that is, $f(t, y)$ is actually only a function of y.
(2) The equation is linear.
(3) The initial value problem $y' = f(t, y)$, $y(0) = y_0$ has a unique solution for all y_0 between -2 and 2.



Solution: (1) The equation is not autonomous, because the derivative y' depends on t
(2) The equation is not linear: if it where, we would have $y' = -p(t)y + q(t)$, and for fixed t the could be only one critical point (which is not true; for example, along $t = 2$ we have critical points at $y = 0$ and $y = 2$).
(3) There are no points $(0, y_0)$ at which the function value (slope shown in the direction field) is not continuously changing from the previous value, and the change (derivative f_y) is similarly continuous, so there should be a unique solution for all of these points.

b. [6 points] Let $y' = g(t, y) = y(y^3 - a^3)$, where a is a real number. Identify all a for which it is true both there is a critical point other than $y = 0$, and that $y = 0$ is stable. Be sure it is clear how you arrive at your conclusion. Draw a phase line for this situation, or explain why it is impossible.

Solution: If $a = 0$, we have $y' = y^4$, and the only critical point is $y = 0$. Note that in this case $y' > 0$ for all $y \neq 0$, so the critical point is unstable (or, if one likes the term, semi-stable). If $a \neq 0$, there are two critical points, $y = 0$ and $y = a$. If $a < 0$, the equation is $y' = y(y^3 + |a|^3)$. For $y > 0$, $y' > 0$; for $-|a| < y < 0$, $y' < 0$, and for $y < -|a|$ $y' > 0$. Thus when $a < 0$, the critical point $y = 0$ is unstable and $y = -|a|$ is stable. Similarly, if $a > 0$, $y' > 0$ when $y < 0$; $y' < 0$ when $0 < y < a$; and $y' > 0$ when $y > a$. Thus $y = 0$ is stable and $y = a$ is unstable. The values of a we want are $a > 0$. The phase line is shown below.

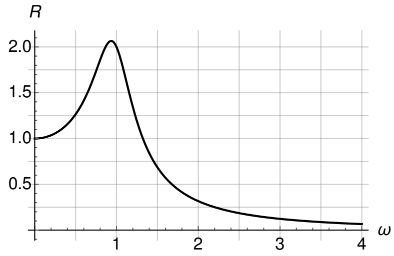


4. [12 points] In lab 4 we consider a forced electrical system of the form

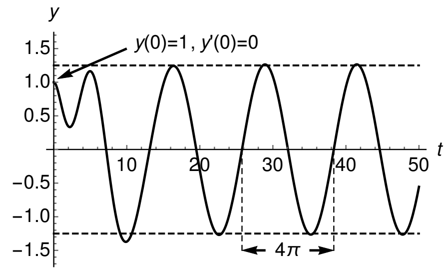
$$y'' + 2\gamma y' + \omega_0^2 y = F(t),$$

which models the current in a circuit. In this problem we take $\gamma = 0.25$ and $\omega_0 = 1$.

a. [6 points] The amplitude of the long-term response to a forcing voltage $F = \sin(\omega t)$ is shown as a function of ω in the figure to the right. Without solving the differential equation, sketch a reasonably accurate graph of the solution to the differential equation with initial conditions $y(0) = 1$, $y'(0) = 0$, when $\omega = 0.5$. (Your graph should include magnitudes and timescales where it is possible to determine them.)



Solution: We know that the solution to the problem will be $y = y_c + y_p$, and with small damping y_c will consist of decaying sinusoidal terms. From the method of undetermined coefficients, y_p will be $y_p = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \phi)$. Thus the steady state response is a pure sinusoidal solution with period $T = \frac{2\pi}{\omega} = 4\pi$, and the amplitude $R(\omega) = R(0.5) = 1.25$, from the graph of the gain function. Given this and the initial conditions, we have the graph below.

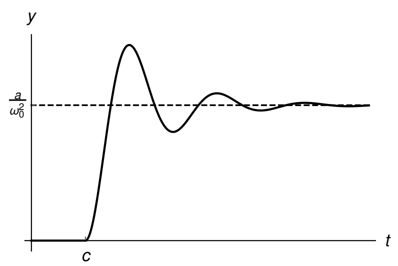


b. [6 points] Now suppose that

$$F(t) = I(t) = \begin{cases} 0, & t < c \\ k, & c \le t, \end{cases}$$

and that $y(0) = y'(0) = 0$. Without solving the differential equation, sketch the behavior of the solution, indicating the magnitude of the solution and its other characteristics as possible.

Solution: For $t < c$, we know the solution will be zero. At $t = c$ we are solving $y'' + 2\gamma y' + \omega_0^2 y = k$, with the initial condition $y(0) = y'(0) = 0$. We know that this will have a decaying oscillatory component, and will converge to a constant steady state $y_p = k/\omega_0^2 = k$. Thus we have the graph shown below.



5. [12 points] Consider the systems model of a linear oscillator given by

$$x' = y, \quad y' = -2x - 2y + k\delta(t - t_0),$$

with initial conditions $x(0) = 0$, $y(0) = 2$.

a. [5 points] **Use Laplace transforms on the system** to find $x(t)$.

Solution: With $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$, the Laplace transform of the first equation in the system is $sX = Y$. The second transforms similarly to give $sY - 2 = -2X - 2Y + ke^{-t_0 s}$. Using $Y = sX$, we can rewrite the second as $s^2X + 2sX + 2X = 2 + ke^{-t_0 s}$, so that

$$X = \frac{2}{(s+1)^2 + 1} + \frac{ke^{-t_0 s}}{(s+1)^2 + 1}.$$

Inverting, we have

$$x(t) = 2e^{-t}\sin(t) + ke^{-(t-t_0)}\sin(t - t_0)u_{t_0}(t).$$

b. [5 points] For what t_0 and k will $x(t)$ be identically zero for all $t > t_0$?

Solution: We must take t_0 to be a multiple of the period (or half-period), so that $y = 0$ at that instant. The period is 2π. Thus, we can take $t = 2n\pi$ (or $t = n\pi$), for n a positive integer. At that point we need k to be the negative of the magnitude of the decaying oscillation $2e^{-t}\sin(t)$: that is, $k = -2e^{-2n\pi}$. (Or, for a half-period, $k = 2e^{-n\pi}$, because the sine term is becoming negative there.)

c. [2 points] Give a physical system that this could model and explain what the result in (b) corresponds to in the model.

Solution: One possible model is that x is the position of a mass on a spring; another is that it is the current in a circuit. In either case, we start with an initial velocity (or voltage) of 2 and no initial displacement (or current). The impulse is like an instantaneous force on the mass that induces a change in momentum to stop all motion (or, an instantaneous change in voltage across the capacitor to zero out the current).

6. [12 points] Consider the nonlinear system

$$x' = 1 - y, \qquad y' = 2 - 2y + 3\sin(x).$$

Sketch a qualitatively accurate phase portrait showing representative trajectories, by doing appropriate linearization and local analysis. Use your phase portrait to predict the behavior of a trajectory starting at $x(0) = \pi$, $y(0) = 0$.

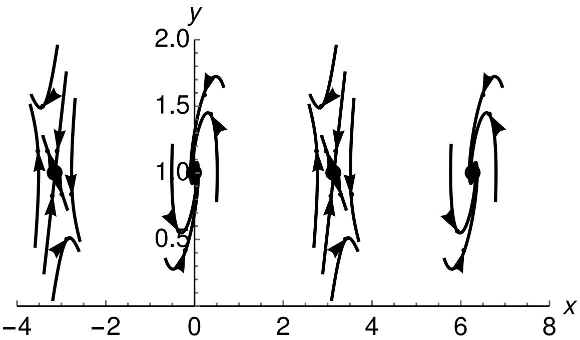
Solution: To sketch the phase portrait, we first find critical points. Requiring $x' = 0 = 1 - y$, we have $y = 1$. Then $y' = 0 = 2 - 2(1) + 3\sin(x)$, so $x = n\pi$, for any integer n. To determine the linear behavior at each of the critical points $(n\pi, 1)$, we find the Jacobian to tell us the coefficient matrices for the linear systems. We have $J(x, y) = \begin{pmatrix} 0 & -1 \\ 3\cos(x) & -2 \end{pmatrix}$. Thus, at $(n\pi, 1)$ with n even and odd, the coefficient matrices are, respectively,

$$J_{2n} = \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix} \quad \text{and} \quad J_{2n+1} = \begin{pmatrix} 0 & -1 \\ -3 & -2 \end{pmatrix}.$$

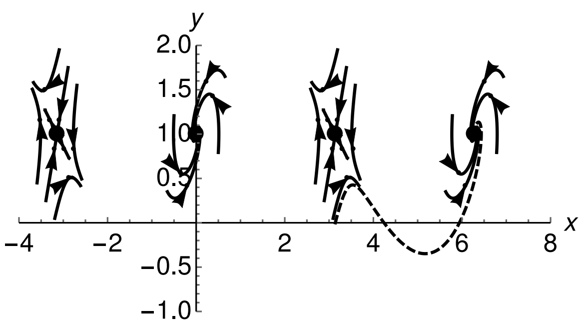
At $x = n\pi$, n even, eigenvalues of the coefficient matrix satisfy $\lambda^2 + 2\lambda + 3 = (\lambda + 1)^2 + 2 = 0$, so $\lambda = -1 \pm i\sqrt{2}$. Thus at these critical points we have a spiral sink. Further, starting from $(1, 0)$ then gives $(x', y') = (0, 3)$, so motion is counter-clockwise around the critical points.

At $x = n\pi$, n odd, eigenvalues of the coefficient matrix satisfy $\lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1) = 0$, so $\lambda = -3$ or $\lambda = 1$, and these are unstable saddle points. The eigenvectors associated with the eigenvalues are, respectively, $\mathbf{v}_{-3} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Putting these together, we have the phase portrait shown below.



From this, we see that an initial condition $(\pi, 0)$ will follow the saddle trajectory below the critical point $(\pi, 1)$ up and to the right, then down, and will likely spiral in to either the critical point $(2\pi, 1)$. This gives the dashed trajectory shown below.



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7. [15 points] For each of the following find explicit, real-valued solutions as indicated. For this problem **do NOT use Laplace transforms**. Note that you do not need to simplify numeric expressions.

a. [8 points] Find the solution to $y' = \dfrac{8 + 4s^2 - 2sy}{2 + s^2}$, $y(0) = 4$.

Solution: This problem is first order and linear, but not separable, so we use an integrating factor for it. Rewriting it in standard form, we have

$$y' + \frac{2s}{2 + s^2}\, y = 4,$$

so that an integrating factor is $\mu = e^{\int 2s/(2+s^2)\,ds} = e^{\ln|2+s^2|} = 2 + s^2$. Multiplying through by μ, we have

$$(2 + s^2)(y' + \frac{2s}{2 + s^2}\, y) = ((2 + s^2)y)' = 4(2 + s^2).$$

Integrating both sides,

$$(2 + s^2)y = 4(2s + \frac{1}{3}s^3) + C,$$

and the initial condition requires that $C = 8$. Thus

$$y = 4\frac{2s + \frac{1}{3}s^3}{2 + s^2} + \frac{8}{2 + s^2}.$$

b. [7 points] Find the general solution to $y'' - 4y' - 5y = -10t + 22$.

Solution: This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be $y = y_c + y_p$, where y_c is the general solution to the complementary homogeneous problem. For this we look for a solution $y = e^{\lambda t}$. Plugging in to the homogeneous equation, we have

$$2\lambda^2 - 8\lambda - 10 = 2(\lambda + 1)(\lambda - 5) = 0.$$

Thus $\lambda = -1$ or $\lambda = 5$, so that y_c is given by $y_c = c_1 e^{-t} + c_2 e^{5t}$. Then, to find y_p, we use undertermined coefficients and look for a solution of the form $y_p = At + B$. Plugging into the differential equation, we have

$$-8A - 10(At + B) = -20t + 44,$$

so that $A = 2$, and $B = -6$, and so

$$y = c_1 e^{-t} + c_2 e^{5t} + 2t - 6.$$

You SHOULD NOT complete this page or the one preceding it if you have completed the mastery assessment.

8. [15 points] For each of the following find explicit, real-valued solutions as indicated.

a. [8 points] Solve $\mathbf{x}' = \begin{pmatrix} -4 & -10 \\ 10 & 8 \end{pmatrix} \mathbf{x}$.

Solution: We use the eigenvalue method to solve the system. We guess $\mathbf{x} = \mathbf{v}e^{\lambda t}$, so that, plugging into the system, $\lambda\mathbf{v} = \mathbf{A}\mathbf{v}$, and λ and $\mathbf{v}$ must be the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Thus

$$\det(\begin{pmatrix} -4-\lambda & -10 \\ 10 & 8-\lambda \end{pmatrix}) = (-4-\lambda)(8-\lambda) + 100 = \lambda^2 - 4\lambda + 68 = (\lambda-2)^2 + 64 = 0,$$

and $\lambda = 2 \pm 8i$. With $\lambda = 2 + 8i$, the eigenvector satisfies $\begin{pmatrix} -6-8i & -10 \\ 10 & 6-8i \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that an eigenvector is a multiple of $\mathbf{v} = \begin{pmatrix} 5 \\ -3-4i \end{pmatrix}$. To find two linearly independent real-valued solutions, we separate the real and imaginary parts of $\mathbf{x} = \mathbf{v}e^{\lambda t} = \mathbf{v}e^{2t}(\cos(8t) + i\sin(8t))$. Using these as our fundamental solution set, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 5\cos(8t) \\ -3\cos(8t) + 4\sin(8t) \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 5\sin(8t) \\ -4\cos(8t) - 3\sin(8t) \end{pmatrix} e^{2t}.$$

b. [7 points] Find the solution to $z'' + 4z' + 20z = 8u_6(t)$, $z(0) = 0$, $z'(0) = 9$.

Solution: Taking the Laplace transform of both sides of the equation, we have, with $Z(s) = \mathcal{L}\{z(t)\}$,

$$s^2 Z - 9 + 4sZ + 20Z = \frac{8e^{-6s}}{s}.$$

Solving for Z, we have $Z(s) = \frac{9}{s^2+4s+20} + \frac{8e^{-6s}}{s(s^2+4s+20)}$. Noting that $s^2 + 4s + 20 = (s+2)^2 + 4^2$, we can invert the first term and can use partial fractions on the second. For the second, we have

$$\frac{8}{s(s^2+4s+20)} = \frac{A}{s} + \frac{B(s+2)+C}{(s+2)^2+4^2},$$

so that, clearing the denominators,

$$8 = A((s+2)^2 + 4^2) + Bs(s+2) + Cs.$$

Letting $s = 0$, we get $A = \frac{2}{5}$. Then from the coefficient of s^2, $B = -A = -\frac{2}{5}$, and finally, from the coefficient of s, $C = -4A - 2B = -\frac{4}{5}$. Thus, inverting the transform, we have

$$\begin{aligned} z(t) &= \mathcal{L}^{-1}\{\frac{9}{(s+2)^2+4^2} + \frac{2}{5}s - \frac{2}{5}\frac{s+2}{(s+2)^2+4^2} + \frac{4}{5}\frac{1}{(s+2)^2+4^2} \\ &= \frac{9}{4}e^{-2t}\cos(4t) + \left(\frac{2}{5} - \frac{2}{5}e^{-2(t-6)}\cos(4(t-6)) - \frac{1}{5}e^{-2(t-6)}\sin(4(t-6))\right)u_6(t). \end{aligned}$$

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\frac{s}{s^2 + a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$
F.	$\int_0^t f(x) g(t-x)\,dx$	$F(s)G(s)$