# Math 216 — First Midterm

17 October, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

**1**. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

**a**. [8 points] Find the solution to the initial value problem $\dfrac{y'}{x^3+y} = \dfrac{1}{x}$, $y(1) = 2$.

**b**. [7 points] Find the general solution to $y' + \frac{1}{t}\,y = \frac{1}{ty}$.

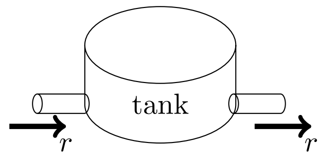
**2**. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

**a**. [8 points] Find the solution to the initial value problem $x' = x + 2y$, $y' = 4x + 3y$, $x(0) = -1$, $y(0) = 8$.[1]

**b**. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

[1]The original exam copy had $y'(0) = 8$; a correct solution may be obtained applying this as well.

**3**. [14 points] Consider a storage tank containing $V_0$ liters of pure water, having an open top, as suggested in the figure to the right. Water containing a chemical at a concentration of $c_0$ kg/liter enters the tank at a rate $r$ liters/min. The well-mixed solution leaves the tank at the same rate.




**a**. [5 points] Write down an initial value problem for the amount $A$ of the chemical $P$ in the tank.

**b**. [5 points] Now suppose that the liquid can evaporate from the top of the tank. This results in a loss proportional to the surface area, so the volume of liquid in the tank decreases by $\alpha\pi a^2$ liters/min, where $a$ is the radius of the (cylindrical) tank. Write (but do not solve) a new differential equation for the amount of chemical in the tank. You should assume that the chemical does not evaporate as well. Be sure that it is clear why your equation has the form it does.

**c**. [4 points] Consider your differential equation in (b) with the initial condition $A(0) = c_0V_0$. On what range of $t$ values, if any, is a unique solution for $A$ guaranteed to exist? Explain.

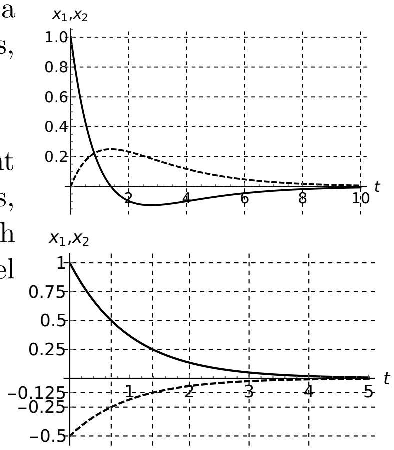
**4**. [15 points] In lab 2 we considered the van der Pol equation, $x'' + \mu(x^2 - 1)x' + x = 0$. We consider this equation in this problem.

**a**. [4 points] Write the van der Pol equation as a system of two first-order differential equations and show that the only critical point of the system is $(0, 0)$.

**b**. [6 points] Let $\mu = 1$. As we saw in lab, the linearization of the system you obtained in (a) at the critical point is then $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{x}$. Solve this system and sketch a phase portrait for it.

**c**. [5 points] Explain what your solution in (b) tells about solutions to the original van der Pol equation. Then, using your system from (a), find the slope of the trajectory in the phase plane at $(3, -\frac{3}{8})$. Explain what these tell you about how the phase portrait for (a) is different from that for (b), and how this is related to your work in lab.

**5**. [15 points] The following considers the solution $(x_1, x_2)$ to a linear system of two first-order constant coefficient equations, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.




**a**. [5 points] If the solutions to this system for two different initial conditions are shown to the right (in both graphs, the solid curve is $x_1$ and the dashed curve is $x_2$), sketch the corresponding trajectories in the phase plane. Label each trajectory.

**b**. [5 points] Given your trajectories in (a), give possible values for the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Be sure that it is clear how you obtain your answer.

**c**. [5 points] Sketch a phase portrait for the system given your answer to (b). *(If you were unable to complete (b), assume that your eigenvalues and eigenvectors are* $\lambda = -2$ *with* $\mathbf{v} = \begin{pmatrix}1 & -1\end{pmatrix}^T$ *and* $\lambda = -1$ *with* $\mathbf{v} = \begin{pmatrix}2 & -1\end{pmatrix}^T$*.)*

**6**. [14 points] Consider a chemical reaction in which two chemicals $X$ and $Y$ combine to form a new compound $Z$. We write $X + Y \to Z$. Then the speed of the reaction (that is, the rate at which the compound $Z$ appears) is proportional to product of the concentrations of the compounds $X$ and $Y$. Because one molecule of each of $X$ and $Y$ are used for each molecule of $Z$ that is created, this results in the differential equation

$$\frac{dz}{dt} = \alpha(x_0 - z)(y_0 - z),$$

where $z$ is the concentration of $Z$, $\alpha$ is the rate constant for the reaction and $x_0$ and $y_0$ are the initial concentrations of $X$ and $Y$. If we initially have none of compound $Z$, the initial condition is $z(0) = 0$.

**a**. [7 points] Suppose that $0 < \alpha < 1$ and $0 < x_0 < y_0$. Without solving the equation, determine what you expect the long-term concentration of $Z$ will be by doing a qualitative analysis of the given equation. (While you may confirm your conclusions by speaking to the chemistry, your answer should be grounded in the analysis of the differential equation.)

**b**. [7 points] Now suppose that $0 < \alpha < 1$ and $x_0 = y_0 > 0$. How does your analysis of the equation from (a) change? Explain by doing a similar analysis.

**7**. [12 points] Each of the following has an answer that you can determine with minimal work. In each, $\mathbf{A}$ is a $2\times 2$ real-valued matrix (but in each is a different matrix). Provide the answer, and give a two sentence explanation of how you obtained it.

**a**. [4 points] If $\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$ and eigenvalues of $\mathbf{A}$ are $\lambda_1$ and $\lambda_2$, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix}3\\4\end{pmatrix}$ is

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**b**. [4 points] If the only eigenvalue of $\mathbf{A}$ is $\lambda = -3$, with only one eigenvector, $\mathbf{v} = \begin{pmatrix}4\\3\end{pmatrix}$, then as $t \to \infty$, the largest term in all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will be

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**c**. [4 points] If the eigenvalues of $\mathbf{A}$ are $\lambda = -3$ and $\lambda = 5$, with eigenvectors $\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}-1\\1\end{pmatrix}$, then the number of solutions $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is

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and the number of solutions to $\mathbf{A}\mathbf{x} = 3\mathbf{x}$ is

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*This page provided for additional work.*