Math 216 — Second Midterm

17 November, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [8 points] Find the solution $z(t)$ to the initial value problem $3z'' + 12z' + 39z = 6e^{-t}$, $z(0) = \frac{1}{5}$, $z'(0) = 0$

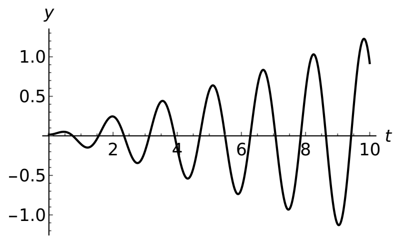
b. [7 points] Find the general solution $y(t)$ to $y'' + 5y' + 6y = \cos(t)$.

2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **USE Laplace transforms**.

a. [8 points] Find the solution $y(t)$ to the initial value problem $y'' + 3y' + 2y = 4$, $y(0) = 0$, $y'(0) = 0$.

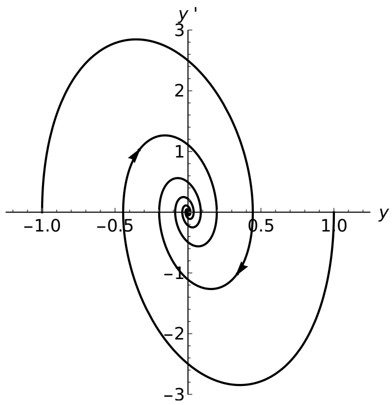
b. [7 points] Find the solution $z(t)$ to the initial value problem $z'' + 2z' + 10z = 0$, $z(0) = 1$, $z'(0) = 3$.

3. [14 points] In this problem we consider the differential equation $y'' + ky' + 16y = F_0 \cos(\omega t)$.



a. [7 points] If the solution to the problem is shown in the figure to the right when $F_0 = 1$, what can you say about the values of k and ω? Solve your equation and explain how your solution would give this graph.

b. [7 points] Now suppose that when $F_0 = 0$ the phase portrait for the equation is shown to the right. Which of $k = -4$, $k = 6$, or $k = 10$ could we have used in this case? Solve the problem with that value of k and explain how your solution would give this graph.



4. [15 points] Note in this problem that $\mathcal{L}$ indicates the Laplace transform.

a. [5 points] If $f(t) = t\,e^{-t}$, use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$.

b. [5 points] Use rules from the table of transforms to confirm your result in (a). Be sure that it is clear what rules you are using and how they give the result you obtain.

c. [5 points] The solution to $y'' + 3y' + 2y = e^{-t}$ with initial conditions $y(0) = 0$, $y'(0) = 2$ is $y = e^{-t} - e^{-2t} + te^{-t}$. Transform the solution to the equation and the equation itself, and show that the two expressions you get for $Y(s) = \mathcal{L}\{y(t)\}$ are the same.

5. [14 points] In lab 3 we considered the nonlinear system

$$N' = \gamma(A - N(1+P)), \qquad P' = P(N-1).$$

We established that the equilibrium solutions to the system are $(N, P) = (A, 0)$ and $(N, P) = (1, A-1)$, and that near the latter the system is approximated by the linear second order problem $v'' + \gamma v' + \gamma A(A-1)v = 0$, where v is the small variation in P from the equilibrium $A - 1$.

a. [4 points] Write the linear, second-order problem from above as a system of two linear, first-order equations.

b. [6 points] Suppose that we pick A and γ so that the characteristic equation of the linear second-order equation has a repeated root. Find the solution to the linear second-order equation in this case, and use your solution to write the solution to the system you found in (a). *(If you are stuck, assume that $A = 2$ and find a nonzero γ to finish the problem with a one point penalty.)*

c. [4 points] In Part B of the lab, we assumed that A was a function of time, that is, $A = A(t) = A_0 + 2a\cos(\omega t)$. Suppose instead we picked $A(t) = A_0 \tan(\omega t)$, so that $v'' + \gamma v' + q(t)v = 0$, with $q(t) = \gamma A(t)(A(t) - 1)$. If we start with $v(0) = 0.5$, $v'(0) = 0$, what is the longest interval on which the solution to the initial value problem is certain to have a unique solution, and why? (Note that you cannot solve the equation by hand.)

6. [15 points] Consider a physical system modeled by the differential equation

$$x'' + \gamma x' + kx = f(t),$$

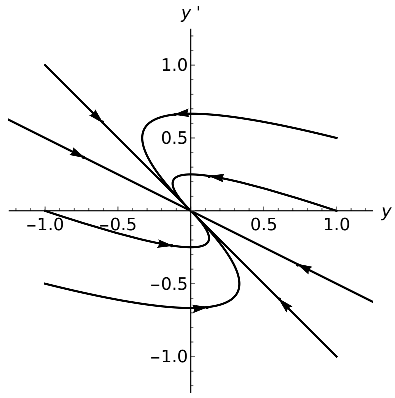
where $x(t)$ is the physical quantity being measured and γ and k are constants.

a. [4 points] If the physical system is underdamped, what can you say about the parameters γ and k?

b. [5 points] If $x(0) = x_0$, $x'(0) = v_0$, and $\mathcal{L}\{f(t)\} = F(s)$, find the transform $X(s) = \mathcal{L}\{x(t)\}$.

c. [6 points] If $f(t) = 0$, assuming as in (a) that the system is underdamped, invert your transform from (b) to find $x(t)$. *(If you are stuck, assume the equation is $x'' + \gamma x' + \gamma^2 x = 0$.)*

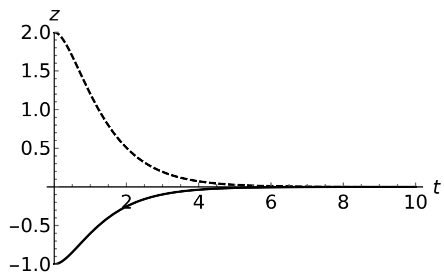
7. [12 points] In the following we consider two linear, homogeneous, second-order, constant coefficient differential equations, for $y(t)$ and $z(t)$. The phase portrait for the equation for $y(t)$ is shown to the right, and graphs of $z(t)$ for two different initial conditions are shown in the figure to the right, below. Explain in a sentence or two why each of the following **cannot** be true.



a. [3 points] The equation is $y'' - 3y' + 2y = 0$

b. [3 points] The general solution to the equation is $y = c_1e^{-t} + c_2e^{-2t}$.

c. [3 points] Given some initial conditions, the Laplace transform $Z(s) = \mathcal{L}\{z(t)\} = \frac{2s+4}{s^2+2s+5}$.



d. [3 points] Written as a system, the equation for $z(t)$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
- A coarse summary of partial fractions:

$$\frac{1}{(s+r_1)(s+r_2)^2((s+h)^2+k^2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{(s+r_2)^2} + \frac{D(s+h)+E}{(s+h)^2+k^2}.$$

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$