Math 216 — Final Exam

17 December, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Consider the system of differential equations $\mathbf{x}' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -4 \end{pmatrix} \mathbf{x}$.

a. [6 points] Find the general solution to this system.[1]

b. [6 points] Now suppose that we consider only initial conditions in the yz-plane (that is, we take $\mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \\ z_0 \end{pmatrix}$). Sketch the phase portrait for these initial conditions, in the yz-plane.

[1] Possibly useful: $\det(\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}) = a(be - cd)$.

2. [12 points] Consider the mass-spring model $2y'' + 4y' + 6y = 8\cos(3t)$.

a. [4 points] Explain why the following statement is true or false: "For any initial condition, the long-term behavior of the mass is the same."

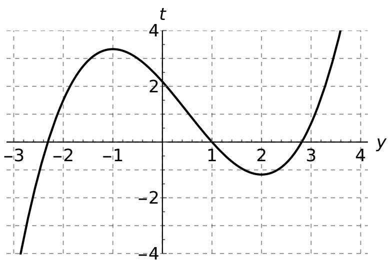
b. [4 points] Explain why the following statement is true or false: "The Laplace transform of the steady state solution to the problem is $\mathcal{L}\{y_{ss}\} = \dfrac{8s}{(2s^2 + 4s + 6)(s^2 + 9)}$."

c. [4 points] Explain why the following statement is true or false: "If we change the forcing term to $f(t) = 8\delta(t - 7)$, the solution y will have a discontinuity at $t = 7$."

3. [10 points] Consider the initial value problem $y' = (y+1)^{-1}(y-2)^{-1}$, $y(0) = 1$.

a. [4 points] Solve your differential equation to find an implicit solution for y, of the form $t = f(y)$.

b. [6 points] Suppose that the graph of the $t(y)$ that you found in (a) is shown to the right. Explain what this tells you about the domain (in t) on which the solution to the initial value problem exists, and how that is related to the theory of first-order equations.



4. [12 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a real-valued 2×2 matrix and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

a. [6 points] Suppose that the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda = -1 \pm i$, with $\mathbf{v} = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$. If $\mathbf{x}$ solves $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, sketch the trajectory for $\mathbf{x}$ in the phase plane.

b. [6 points] Suppose that eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $\mathbf{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, as $t \to \infty$, which of the following is most correct, and why? (i) $x_2 \approx 2x_1$; (ii) $x_2 \approx -\frac{1}{2}x_1$; (iii) $x_2 \approx -\frac{1}{2}x_1 - 1$; (iv) $x_2 \approx -\frac{1}{2}x_1 - k$, with $k > 1$.

5. [12 points] In lab 6 we considered the Fitzhugh-Nagumo model for the behavior of a neuron,

$$v' = v - \frac{1}{3}v^3 - w + I_{ext}, \qquad \tau w' = v + a - bw.$$

In this problem we analyze this with the parameters $\tau = 1$, $a = \frac{1}{3}$, and $b = 1$.

a. [3 points] Find the v- and w-nullclines, and show that there is a single critical point (v_c, w_c) in this case. Find the critical point in terms of the externally applied voltage I_{ext}.

b. [3 points] Linearize the system at the critical point (write your linearization in terms of v_c and w_c—do not plug in the values you found for v_c and w_c). How is the solution to your linearized system related to the solution of the original nonlinear system?

c. [6 points] Show that the critical point is in this case is always stable. Determine any values of v_c or w_c at which the behavior at the critical point changes. Explain how this result is different from that which you saw in lab.

6. [12 points] Consider the nonlinear system

$$x' = 3x - y - x^2, \qquad y' = -\alpha + x - y,$$

where α is a real-valued parameter.

a. [4 points] Find all critical points for the system, and show that for $\alpha > -1$ there are two critical points, if $\alpha = -1$ there is one, and if $\alpha < -1$ there are none.

b. [8 points] Let $\alpha = 0$: then the system has two critical points, $(0, 0)$ and $(2, 2)$. Sketch a phase portrait for the nonlinear system by linearizing at critical points and determining the resulting behavior in the phase plane.

7. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment.
Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the solution to the initial value problem $W' = \dfrac{-W + 5t}{2}$, $W(0) = 6$.

b. [8 points] Find the general solution to the system of first-order linear differential equations, $x' = -8x - y$, $y' = 45x + 4y$.

8. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment.
Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the general solution $Q(t)$ to the differential equation $Q''(t) - 2Q'(t) + 10Q(t) = 30t$.

b. [8 points] Find the solution to the inital value problem $v''(t) - 8v'(t) + 25v(t) = 3u_7(t)$, $v(0) = 0$, $v'(0) = 6$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$. Thus $\int t\,e^t\,dt = t\,e^t - e^t + C$; $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$; and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
- A coarse summary of partial fractions:

$$\frac{1}{(s+r_1)(s+r_2)^2((s+h)^2+k^2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{(s+r_2)^2} + \frac{D(s+h)+E}{(s+h)^2+k^2}.$$

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t}f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$
F.	$\int_0^t f(x)g(t-x)\,dx$	$F(s)G(s)$