Math 216 — First Midterm

17 October, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

a. [8 points] Find the solution to the initial value problem $\dfrac{y'}{x^3+y} = \dfrac{1}{x}$, $y(1) = 2$.

Solution: This is a linear problem, in standard form $y' - \frac{1}{x}\,y = x^2$. The integrating factor is $\mu = e^{-\int \frac{1}{x}\,dx} = x^{-1}$, so after multiplying through by μ we have $(\frac{1}{x}\,y)' = x$. Integrating and multiplying through by x gives $y = \frac{1}{2}x^3 + Cx$, so that with $y(1) = 2$ we have $C = \frac{3}{2}$, and

$$y = \frac{1}{2}x^3 + \frac{3}{2}x.$$

b. [7 points] Find the general solution to $y' + \frac{1}{t}\,y = \frac{1}{ty}$.

Solution: This is not linear, but can be separated. We have $y' = \frac{1}{t}(\frac{1}{y} - y)$, so that $\dfrac{y'}{\frac{1}{y} - y} = \dfrac{1}{t}$, or $\dfrac{yy'}{1-y^2} = \dfrac{1}{t}$. We are able to integrate both sides, finding $-\frac{1}{2}\ln|1-y^2| = \ln|t| + C'$. Multiplying by -2 and using rules of logs, this is $\ln|1-y^2| = \ln(|t|^{-2}) + C'$, so that, exponentiating and letting $C = \pm e^{C'}$, $1 - y^2 = Ct^{-2}$. Thus

$$y = \pm\sqrt{1 - Ct^{-2}}.$$

2. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

a. [8 points] Find the solution to the initial value problem $x' = x + 2y$, $y' = 4x + 3y$, $x(0) = -1$, $y(0) = 8$.[1]

Solution: In matrix form, this is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. The eigenvalues of the coefficient matrix are given by $\det(\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}) = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1) = 0$. Thus $\lambda = 5$ or $\lambda = -1$. If $\lambda = 5$, eigenvectors satisfy $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Similarly, if $\lambda = -1$, eigenvectors satisfy $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

Applying the initial conditions, we have $c_1 - c_2 = -1$ and $2c_1 + c_2 = 8$. Adding the two equations, $3c_1 = 7$, so $c_1 = \frac{7}{3}$ and $c_2 = \frac{10}{3}$. Thus

$$\mathbf{x} = \frac{7}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + \frac{10}{3} \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-t}.$$

b. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: The eigenvalues of the coefficient matrix are given by $(2-\lambda)(-\lambda) + 2 = \lambda^2 - 2\lambda + 2 = (\lambda-1)^2 + 1 = 0$. Thus $\lambda = 1 \pm i$. If $\lambda = 1 + i$, the components of the eigenvector satisfy (from the second row of the coefficient matrix) $v_1 - (1+i)v_2 = 0$, so we may take $\mathbf{v} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$. A complex-valued solution is therefore $\mathbf{x} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^t(\cos(t) + i\sin(t))$. Separating the real and imaginary parts of this, we have

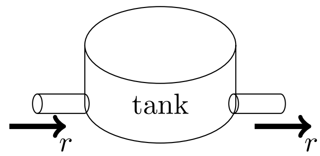
$$\mathbf{x} = c_1 \begin{pmatrix} \cos(t) - \sin(t) \\ \cos(2t) \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{pmatrix} e^t.$$

Alternately, we could take $\mathbf{v} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$, so that

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) \\ \cos(t) + \sin(t) \end{pmatrix} e^t + c_2 \begin{pmatrix} 2\sin(t) \\ -\cos(t) + \sin(t) \end{pmatrix} e^t.$$

[1] The original exam copy had $y'(0) = 8$; a correct solution may be obtained applying this as well.

3. [14 points] Consider a storage tank containing V_0 liters of pure water, having an open top, as suggested in the figure to the right. Water containing a chemical at a concentration of c_0 kg/liter enters the tank at a rate r liters/min. The well-mixed solution leaves the tank at the same rate.



a. [5 points] Write down an initial value problem for the amount A of the chemical P in the tank.

Solution: We have $A' =$ (rate in) − (rate out). The rate in is c_0 kg/liter $\cdot\, r$ liter/min; the rate out is (A/V_0) kg/liter $\cdot\, r$ liter/min. Thus $A' = c_0 r - \frac{r}{V_0} A$. The initial condition is that the tank initially does not contain any of the chemical, so $A(0) = 0$.

b. [5 points] Now suppose that the liquid can evaporate from the top of the tank. This results in a loss proportional to the surface area, so the volume of liquid in the tank decreases by $\alpha\pi a^2$ liters/min, where a is the radius of the (cylindrical) tank. Write (but do not solve) a new differential equation for the amount of chemical in the tank. You should assume that the chemical does not evaporate as well. Be sure that it is clear why your equation has the form it does.

Solution: Note that the volume in the tank is now not constant. Because the evaporation rate does not depend on the volume, and the tank is assumed cylindrical, the loss rate does not change with time, and we have $V = V_0 - \alpha\pi a^2\, t$. Thus our new equation is

$$A' = c_0 r - \frac{r}{V_0 - \alpha\pi a^2\, t}\, A.$$

c. [4 points] Consider your differential equation in (b) with the initial condition $A(0) = c_0 V_0$. On what range of t values, if any, is a unique solution for A guaranteed to exist? Explain.

Solution: We note that the equation we have is linear, so we are guaranteed by our existence and uniqueness theorems that we will have a unique solution whereever the coefficients are continuous. This will be the case for $0 \le t < \frac{V_0}{\alpha\pi a^2}$.

4. [15 points] In lab 2 we considered the van der Pol equation, $x'' + \mu(x^2 - 1)x' + x = 0$. We consider this equation in this problem.

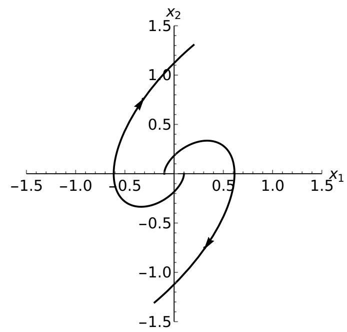
a. [4 points] Write the van der Pol equation as a system of two first-order differential equations and show that the only critical point of the system is $(0,0)$.

Solution: Letting $x_1 = x$ and $x_2 = x'$, we have

$$x_1' = x_2,$$
$$x_2' = -x_1 - \mu(x_1^2 - 1)x_2.$$

Critical points occur when $x_1' = x_2' = 0$. If $x_1' = 0$, we know that $x_2 = 0$. Then the second equation gives $x_2' = -x_1 = 0$, so the only critical point is $(0,0)$.

b. [6 points] Let $\mu = 1$. As we saw in lab, the linearization of the system you obtained in (a) at the critical point is then $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{x}$. Solve this system and sketch a phase portrait for it.



Solution: Let $\mathbf{x} = \mathbf{v}e^{\lambda t}$. The λ are the eigenvalues of the coefficient matrix, which satisfy $\lambda^2 - \mu\lambda + 1 = 0$, so $\lambda = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}$. Because $0 < \mu < 2$, this is $\lambda = \frac{\mu}{2} \pm \frac{1}{2}i\sqrt{4 - \mu^2}$, and trajectories in the phase plane are outward spirals. Noting that at $(1,0)$, $x_1' = 0$ and $x_2' = -1$, we conclude that they are clockwise spirals, as shown in the figure to the right.

Completing the solution of the system, note that the first row of the equation for $\mathbf{v}$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, gives $\mathbf{v} = \begin{pmatrix} 1 & \lambda \end{pmatrix}^T$. Separating the real and imaginary parts of the solution $\mathbf{x} = \mathbf{v}e^{\lambda t}$ allows us to write, with $\omega = \frac{1}{2}\sqrt{4 - \mu^2}$,
$$\mathbf{x} = c_1 \begin{pmatrix} \cos(\omega t) \\ \frac{\mu}{2}\cos(\omega t) - \omega \sin(\omega t) \end{pmatrix} e^{\mu t/2} + c_2 \begin{pmatrix} \sin(\omega t) \\ \omega \cos(\omega t) + \frac{\mu}{2}\sin(\omega t) \end{pmatrix} e^{\mu t/2}.$$

c. [5 points] Explain what your solution in (b) tells about solutions to the original van der Pol equation. Then, using your system from (a), find the slope of the trajectory in the phase plane at $(3, -\frac{3}{8})$. Explain what these tell you about how the phase portrait for (a) is different from that for (b), and how this is related to your work in lab.

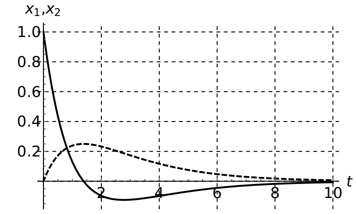
Solution: Because this is a linearization at $(0,0)$, we expect the nonlinear van der Pol equation to have this behavior when x is small. That is, if we start with a small initial condition for x we initially expect the solution to be an oscillatory function with increasing magnitude.

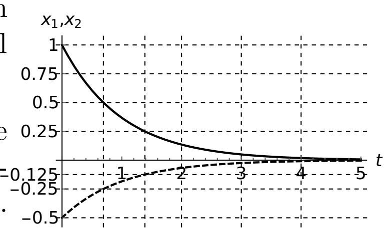
Plugging $(-3, -\frac{3}{8})$ into the nonlinear system from (a), we have $(x_1', x_2') = (-\frac{3}{8}, 0)$. Thus at this point trajectories are moving *inward*. This is different from the linear behavior, and indicates that farther away from $(0,0)$ the nonlinear effects become significant, which results in trajectories from near $(0,0)$ and far from $(0,0)$ converging to the limit cycle that we saw in lab.

5. [15 points] The following considers the solution (x_1, x_2) to a linear system of two first-order constant coefficient equations,

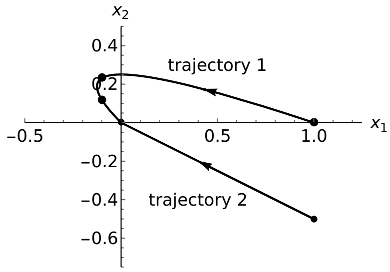
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

a. [5 points] If the solutions to this system for two different initial conditions are shown to the right (in both graphs, the solid curve is x_1 and the dashed curve is x_2), sketch the corresponding trajectories in the phase plane. Label each trajectory.





Solution: For the first trajectory, we note that we start at $(1, 0)$ in the phase plane. At $t = 2$ we are approximately at $(-0.1, 0.2)$, and at $t = 4$, at $(-0.1, 0.1)$. Drawing a smooth curve through these, estimating more points from the graph shown, we get trajectory 1 in the lower graph (which shows the points at $t = 0$, $t = 2$, and $t = 4$ as well). For the second trajectory the provided t values are irregular, but give the points $(1, -0.5)$, $(0.5, -0.25)$, and $(0.25, -0.125)$. Thus, this appears to be a linear trajectory with slope $m = -0.5$, the lower trajectory shown in the figure to the right.



b. [5 points] Given your trajectories in (a), give possible values for the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Be sure that it is clear how you obtain your answer.

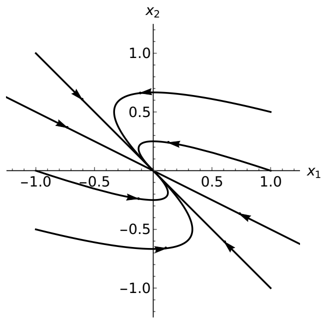
Solution: The linear trajectory tells us one eigenvector: $\mathbf{v}_1 = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$, and we know that the associated eigenvalue must be negative. Because we have a second trajectory that collapses to the origin both eigenvalues must be negative, and all eigenvalues and eigenvectors must be real. The values of the eigenvalues, their relative magnitude, and the second eigenvector (if any) are not definitively specified given the accuracy that we are able to produce in our graph above, though if there is a second eigenvector it clearly must have negative slope to avoid intersecting trajectory 1.

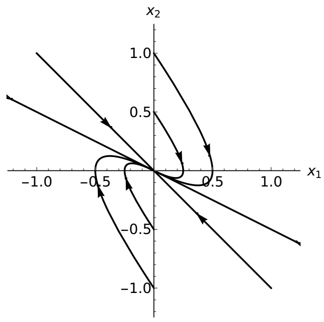
With some thought we might conclude that it must have a slope close to or less (more negative) than the first to avoid an intersection with trajectory 1 as $t \to -\infty$, but this is a subtle point—at the expected accuracy of the figure generated in (a) this observation may be difficult to make.

If there is a second eigenvector with slope less than $m = -\frac{1}{2}$, we could take $\mathbf{v}_2 = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. In this case trajectory 1 in the graph above requires that the eigenvalue associated with $\mathbf{v}_1$ be less than that associated with $\mathbf{v}_2$, e.g., $\lambda_1 = -2$ and $\lambda_2 = -1$. (If we picked $\mathbf{v}_2$ to have slope slightly larger than $m = -\frac{1}{2}$ the order of the eigenvalues would have to reverse for the trajectory to behave as shown.)

c. [5 points] Sketch a phase portrait for the system given your answer to (b). *(If you were unable to complete (b), assume that your eigenvalues and eigenvectors are* $\lambda = -2$ *with* $\mathbf{v} = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$ *and* $\lambda = -1$ *with* $\mathbf{v} = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$*.)*

Solution: With the eigenvectors indicated above, we have two straight line solutions, $y = -x/2$ and $y = -x$. Trajectories collapse fastest along the first, giving the phase portrait to the left, below. If we used the eigenvalues and eigenvectors in the parenthetical note, trajectories collapse first to the other eigenvector and we would have the figure to the right.





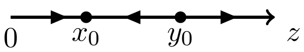
6. [14 points] Consider a chemical reaction in which two chemicals X and Y combine to form a new compound Z. We write $X + Y \to Z$. Then the speed of the reaction (that is, the rate at which the compound Z appears) is proportional to product of the concentrations of the compounds X and Y. Because one molecule of each of X and Y are used for each molecule of Z that is created, this results in the differential equation

$$\frac{dz}{dt} = \alpha(x_0 - z)(y_0 - z),$$

where z is the concentration of Z, α is the rate constant for the reaction and x_0 and y_0 are the initial concentrations of X and Y. If we initially have none of compound Z, the initial condition is $z(0) = 0$.

a. [7 points] Suppose that $0 < \alpha < 1$ and $0 < x_0 < y_0$. Without solving the equation, determine what you expect the long-term concentration of Z will be by doing a qualitative analysis of the given equation. (While you may confirm your conclusions by speaking to the chemistry, your answer should be grounded in the analysis of the differential equation.)

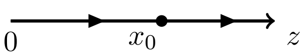
Solution: We see that equilibrium solutions are $z = x_0$ and $z = y_0$. The right-hand side of the equation is an upward opening parabola, so we will have the phase line shown below.



This indicates that the critical point $z = x_0$ is stable, and this is the long-term expected concentration of Z provided $z(0) < y_0$. As the reaction is purported to create Z, we expect $z(0) = 0$, so that $z \to x_0$. (At this point all of the chemical X is used up, so that $x = 0$, and we will have $y = y_0 - x_0$. We note that physically we are unable to create amounts of Z that are greater than either of x_0 or y_0.)

b. [7 points] Now suppose that $0 < \alpha < 1$ and $x_0 = y_0 > 0$. How does your analysis of the equation from (a) change? Explain by doing a similar analysis.

Solution: Now there is a single equilibrium solution, x_0, which is semi-stable (that is, unstable). However, because we do not expect $z > x_0$ at any time, we expect the same long-term behavior: $z \to x_0$. This is illustrated in the phase line, shown below.



7. [12 points] Each of the following has an answer that you can determine with minimal work. In each, $\mathbf{A}$ is a 2×2 real-valued matrix (but in each is a different matrix). Provide the answer, and give a two sentence explanation of how you obtained it.

a. [4 points] If $\mathbf{A}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$ and eigenvalues of $\mathbf{A}$ are λ_1 and λ_2, with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix}3\\4\end{pmatrix}$ is

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} - \begin{pmatrix}1\\2\end{pmatrix}$$

Solution: We see that $\mathbf{x}_c = -\begin{pmatrix}1\\2\end{pmatrix}$ is the equilibrium solution for the given equation. Letting $\mathbf{x} = \mathbf{u} + \mathbf{x}_c$ will result in $\mathbf{u}$ solving the homogeneous equation $\mathbf{u}' = \mathbf{A}\mathbf{u}$, so that $\mathbf{u} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$, which gives the indicated general solution.

b. [4 points] If the only eigenvalue of $\mathbf{A}$ is $\lambda = -3$, with only one eigenvector, $\mathbf{v} = \begin{pmatrix}4\\3\end{pmatrix}$, then as $t \to \infty$, the largest term in all solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will be

$$\mathbf{x} = c\,t\,\mathbf{v}e^{-3t} = c\,t\begin{pmatrix}4\\3\end{pmatrix}e^{-3t}$$

Solution: The general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is $\mathbf{x} = c_1(\mathbf{v}t + \mathbf{w})e^{\lambda t} + c_2\mathbf{v}e^{\lambda t}$. The dominant term in this expression is $\mathbf{x} = c_1\mathbf{v}te^{\lambda t}$.

c. [4 points] If the eigenvalues of $\mathbf{A}$ are $\lambda = -3$ and $\lambda = 5$, with eigenvectors $\mathbf{v} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}-1\\1\end{pmatrix}$, then the number of solutions $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is

Exactly one: $\mathbf{x} = \mathbf{0}$.

and the number of solutions to $\mathbf{A}\mathbf{x} = 3\mathbf{x}$ is

Exactly one.

Solution: Note that if $\mathbf{A}$ doesn't have an eigenvalue $\lambda = 0$, then its determinant must be nonzero, so $\mathbf{A}\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$. Similarly, $\mathbf{A}\mathbf{x} = k\mathbf{x}$ has multiple solutions only if k is an eigenvalue, which isn't the case here.