Math 216 — Second Midterm

17 November, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [8 points] Find the solution $z(t)$ to the initial value problem $3z'' + 12z' + 39z = 6e^{-t}$, $z(0) = \frac{1}{5}$, $z'(0) = 0$

Solution: This problem is nonhomogeneous, linear, and constant-coefficient. The general solution will be $z = z_c + z_p$, where z_c is the general solution to the complementary homogeneous problem. For this we look for a solution $z = e^{\lambda t}$. Plugging in to the homogeneous equation, we have $3\lambda^2 + 12\lambda + 39 = 3(\lambda^2 + 4\lambda + 13) = 3((\lambda + 2)^2 + 9) = 0$. Thus $\lambda = -2 \pm 3i$. Separating the real and imaginary parts of the resulting complex exponential, we have $z_c = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$.

Then, to find z_p, we use undertermined coefficients and look for a solution of the form $z_p = Ae^{-t}$. Plugging into the differential equation, we have $3A - 12A + 39A = 6$, or $10A = 2$, so that $A = \frac{1}{5}$. Thus

$$z = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t) + \frac{1}{5}\,e^{-t}.$$

Applying the initial conditions, we have $z(0) = c_1 + \frac{1}{5} = \frac{1}{5}$, so $c_1 = 0$. Then $z'(0) = 3c_2 - \frac{1}{5} = 0$, so that $c_2 = \frac{1}{15}$, and

$$z = \frac{1}{15}e^{-2t}\sin(3t) + \frac{1}{5}\,e^{-t}.$$

b. [7 points] Find the general solution $y(t)$ to $y'' + 5y' + 6y = \cos(t)$.

Solution: Again, the general solution will be $y = y_c + y_p$. For y_c, the characteristic equation is $\lambda^2+5\lambda+6 = (\lambda+2)(\lambda+3) = 0$, so $\lambda = -2$ or $\lambda = -3$, and $y_c = c_1e^{-2t}+c_2e^{-3t}$.

For y_p, we again use the method of undetermined coefficients and guess $y_p = a\cos(t)+ b\sin(t)$. Plugging into the equation, we have

$$(-a\cos(t) - b\sin(t)) + (-5a\sin(t) + 5b\cos(t)) + 6a\cos(t) + 6b\sin(t) = \cos(t).$$

Collecting the $\cos(t)$ and $\sin(t)$ terms, this is $5a + 5b = 1$ and $-5a + 5b = 0$. Adding the two we get $b = \frac{1}{10}$, so that from the second $a = \frac{1}{10}$, and the

$$y = c_1e^{-2t} + c_2e^{-3t} + \frac{1}{10}\cos(t) + \frac{1}{10}\sin(t).$$

2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **USE Laplace transforms**.

a. [8 points] Find the solution $y(t)$ to the initial value problem $y'' + 3y' + 2y = 4$, $y(0) = 0$, $y'(0) = 0$.

Solution: Taking the Laplace transform of both sides of the equation, we have $\mathcal{L}\{y'' + 3y' + 2y\} = \frac{4}{s}$, so that, with $Y = \mathcal{L}\{y\}$,

$$s^2Y + 3sY + 2Y = \frac{4}{s},$$

so that

$$Y = \frac{4}{s(s^2 + 3s + 2)} = \frac{4}{s(s + 1)(s + 2)}.$$

Partial fractions allows us to rewrite $\frac{4}{s(s+1)(s+2)} = A\frac{1}{s} + B\frac{1}{s+1} + C\frac{1}{s+2}$, so that we can find the inverse transform

$$y(t) = \mathcal{L}^{-1}\{A\frac{1}{s} + B\frac{1}{s + 1} + C\frac{1}{s + 2}\} = A + Be^{-t} + Ce^{-2t}.$$

To find the constants A, B, and C, we solve in the equality giving the partial fractions decomposition. Clearing the denominators, we have $4 = A(s + 1)(s + 2) + Bs(s + 2) + Cs(s + 1)$. Plugging in $s = 0$, $A = 2$; with $s = -1$, $B = -4$; and with $s = -2$, $C = 2$. Thus

$$y = 2 - 4e^{-t} + 2e^{-2t}.$$

b. [7 points] Find the solution $z(t)$ to the initial value problem $z'' + 2z' + 10z = 0$, $z(0) = 1$, $z'(0) = 3$.

Solution: Proceding as above, the forward transform gives

$$(s^2Z - s - 3) + 2(sZ - 1) + 10Z = 0,$$

so that

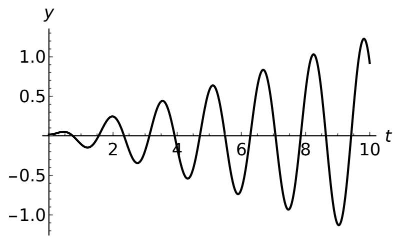
$$Z = \frac{s + 5}{s^2 + 2s + 10} = \frac{s + 5}{(s + 1)^2 + 9}.$$

To find the inverse transform, we rewrite the right hand side as $\frac{s+5}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} + \frac{4}{(s+1)^2+9}$. We can then invert both terms to get

$$z = \mathcal{L}^{-1}\{\frac{s + 1}{(s + 1)^2 + 9}\} + \mathcal{L}^{-1}\{\frac{4}{(s + 1)^2 + 9}\} = e^{-t}\cos(3t) + \frac{4}{3}e^{-t}\sin(3t).$$

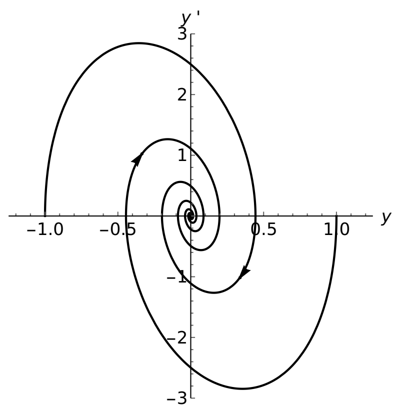
3. [14 points] In this problem we consider the differential equation $y'' + ky' + 16y = F_0 \cos(\omega t)$.

a. [7 points] If the solution to the problem is shown in the figure to the right when $F_0 = 1$, what can you say about the values of k and ω? Solve your equation and explain how your solution would give this graph.



Solution: The simplest guess is that we are seeing forcing of an undamped problem, so that $k = 0$, at the natural frequency of the system, which is $\omega = \sqrt{16} = 4$. In this case we're solving $y'' + 16y = \cos(4t)$. The complementary homogeneous solution is $y_c = c_1 \cos(4t) + c_2 \sin(4t)$. The particular solution will be $y_p = at\cos(4t) + bt\sin(4t)$, and because there are no odd derivatives in the problem we will find $a = 0$. Then $y_p' = b\sin(4t) + 4bt\cos(4t)$ and $y_p'' = 8b\cos(4t) - 16bt\sin(4t)$; plugging in, we get $8b\cos(4t) - 16bt\sin(4t) + 16bt\sin(4t) = \cos(4t)$, so that $b = \frac{1}{8}$. Thus the general solution is $y = c_1\cos(4t) + c_2\sin(4t) + \frac{1}{8}\,t\,\sin(4t)$, which will have a linearly growing solution as shown in the figure.

b. [7 points] Now suppose that when $F_0 = 0$ the phase portrait for the equation is shown to the right. Which of $k = -4$, $k = 6$, or $k = 10$ could we have used in this case? Solve the problem with that value of k and explain how your solution would give this graph.



Solution: We see that the eigenvalues are complex with negative real part. Solving the characteristic equation, we have $\lambda^2 + k\lambda + 16 = 0$, so that $\lambda = -\frac{k}{2} \pm \frac{1}{2}\sqrt{k^2 - 64}$. For this to have complex roots, $-8 < k < 8$, and for the real part to be negative, $k > 0$. Thus we require that $0 < k < 8$, and $k = 6$ is the only option of those provided that works. In this case, the roots of the equation are $\lambda = -3 \pm \frac{1}{2}i\sqrt{28} = -3 \pm i\sqrt{7}$, so that the general solution is $y = c_1 e^{-3t}\cos(\sqrt{7}\,t) + c_2 e^{-3t}\sin(\sqrt{7}\,t)$, which is a decaying oscillatory solution that will have inward spiral trajectories in the phase plane.

Alternately, we could see what the eigenvalues are for each of the indicated values of k: if $k = -4$, $\lambda^2 - 4\lambda + 16 = 0$, and $\lambda = 2 \pm i\frac{1}{2}\sqrt{48} = 2 \pm i2\sqrt{3}$, which would growing spiral solutions. If $k = 6$, $\lambda = -3 \pm i\sqrt{7}$, as shown above. If $k = 10$, $\lambda = -5 \pm 6 = 1, -11$. This will not have oscillatory solutions, and so cannot give the spiral shown.

4. [15 points] Note in this problem that $\mathcal{L}$ indicates the Laplace transform.

a. [5 points] If $f(t) = t\,e^{-t}$, use the integral definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$.

Solution: We have

$$\begin{aligned}\mathcal{L}\{f(t)\} = \int_0^\infty te^{-t}e^{-st}\,dt = \int_0^\infty te^{-(s+1)t}\,dt \\ = -\frac{1}{s+1}te^{-(s+1)t}\Big|_{t=0}^{t\to\infty} + \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\,dt \\ = 0 - \frac{1}{(s+1)^2}e^{-(s+1)t}\Big|_{t=0}^{t\to\infty} = \frac{1}{(s+1)^2}.\end{aligned}$$

b. [5 points] Use rules from the table of transforms to confirm your result in (a). Be sure that it is clear what rules you are using and how they give the result you obtain.

Solution: There are several ways we could approach this. First, we could note that with $g(t) = t$, $\mathcal{L}\{g(t)\} = \frac{1}{s^2} = G(s)$, so that $\mathcal{L}\{e^{-t}\cdot t\} = G(s+1) = \frac{1}{(s+1)^2} = \mathcal{L}\{te^{-t}\}$.

Alternately, we could note that with $g(t) = e^{-t}$, $\mathcal{L}\{g(t)\} = \frac{1}{s+1} = G(s)$. Then $\mathcal{L}\{tg(t)\} = -G'(s) = \frac{1}{(s+1)^2} = \mathcal{L}\{te^{-t}\}$.

c. [5 points] The solution to $y'' + 3y' + 2y = e^{-t}$ with initial conditions $y(0) = 0$, $y'(0) = 2$ is $y = e^{-t} - e^{-2t} + te^{-t}$. Transform the solution to the equation and the equation itself, and show that the two expressions you get for $Y(s) = \mathcal{L}\{y(t)\}$ are the same.

Solution: The transform of the differential equation gives $(-2+s^2Y)+3sY+2Y = \frac{1}{s+1}$, so that

$$Y = \left(\frac{2}{s^2+3s+2}\right) + \left(\frac{1}{(s+1)(s^2+3s+2}\right) = \frac{2(s+1)+1}{(s+1)^2(s+2)} = \frac{2s+3}{(s+1)^2(s+2)}.$$

The transform of the given solution is

$$\begin{aligned}\mathcal{L}\{y\} &= \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{(s+1)^2} \\ &= \frac{(s+1)(s+2) - (s+1)^2 + (s+2)}{(s+1)^2(s+2)} = \frac{2s+3}{(s+1)^2(s+2)},\end{aligned}$$

which is the same.

5. [14 points] In lab 3 we considered the nonlinear system

$$N' = \gamma(A - N(1+P)), \qquad P' = P(N-1).$$

We established that the equilibrium solutions to the system are $(N,P) = (A,0)$ and $(N,P) = (1, A-1)$, and that near the latter the system is approximated by the linear second order problem $v'' + \gamma v' + \gamma A(A-1)v = 0$, where v is the small variation in P from the equilibrium $A-1$.

a. [4 points] Write the linear, second-order problem from above as a system of two linear, first-order equations.

Solution: Let $x = v$ and $y = v'$. Then $x' = y$ and $y' = -\gamma y - \gamma A(A-1)x$, or, if we prefer a matrix formulation, $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\gamma A(A-1) & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

b. [6 points] Suppose that we pick A and γ so that the characteristic equation of the linear second-order equation has a repeated root. Find the solution to the linear second-order equation in this case, and use your solution to write the solution to the system you found in (a). *(If you are stuck, assume that $A = 2$ and find a nonzero γ to finish the problem with a one point penalty.)*

Solution: Note that the characteristic equation of the linear equation is $\lambda^2+\gamma\lambda+\gamma A(A-1) = 0$, so that $\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\gamma A(A-1)}$. If there is a repeated root, the argument of the square root vanishes, so that we are left with $\lambda = -\frac{\gamma}{2}$, repeated. The general solution is then $v = c_1 e^{-\gamma t/2} + c_2 t e^{-\gamma t/2}$. This is just the top, x, component of the solution to the system in (a). The bottom component is its derivative, $y = v' = -\frac{\gamma}{2}\, c_1 e^{-\gamma t/2} + c_2(1 - \frac{\gamma}{2}\, t)e^{-\gamma t/2}$. Thus the solution to the system is just $x = c_1 e^{-\gamma t/2} + c_2 t e^{-\gamma t/2}$, $y = -\frac{\gamma}{2}\, c_1 e^{-\gamma t/2} + c_2(1 - \frac{\gamma}{2}\, t)e^{-\gamma t/2}$, or, in matrix form,

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\frac{\gamma}{2} \end{pmatrix} e^{-\gamma t/2} + c_2 \left(\begin{pmatrix} 1 \\ -\frac{\gamma}{2} \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{-\gamma t/2}.$$

With $A = 2$, we have $\lambda^2 + \gamma\lambda + 2\gamma = 0$, so that $\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 8\gamma} = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma(\gamma-8)}$, so we have a repeated root if $\gamma = 8$ (or $\gamma = 0$, but we do not consider that). In this case $\lambda = -4$, so that $y = c_1 e^{-4t} + c_2 t e^{-4t}$.

c. [4 points] In Part B of the lab, we assumed that A was a function of time, that is, $A = A(t) = A_0 + 2a\cos(\omega t)$. Suppose instead we picked $A(t) = A_0 \tan(\omega t)$, so that $v'' + \gamma v' + q(t)v = 0$, with $q(t) = \gamma A(t)(A(t) - 1)$. If we start with $v(0) = 0.5$, $v'(0) = 0$, what is the longest interval on which the solution to the initial value problem is certain to have a unique solution, and why? (Note that you cannot solve the equation by hand.)

Solution: We know that there will be a unique solution everywhere the coefficients of the equation are continuous. In this case the only problem is where $q(t) = \gamma A_0 \tan(\omega t)(A_0 \tan(\omega t) - 1)$ is discontinuous, which is where $t = \frac{n\pi}{2\omega}$ (for any odd integer n). Thus we are certain of a unique solution for $0 \le t < \frac{\pi}{2\omega}$.

6. [15 points] Consider a physical system modeled by the differential equation

$$x'' + \gamma x' + kx = f(t),$$

where $x(t)$ is the physical quantity being measured and γ and k are constants.

a. [4 points] If the physical system is underdamped, what can you say about the parameters γ and k?

Solution: If the system is underdamped, we know that the roots of the characteristic equation are complex. The characteristic equation is $\lambda^2 + \gamma\lambda + k = 0$, so that $\lambda = \frac{1}{2}(-\gamma \pm \sqrt{\gamma^2 - 4k})$, so we know that $\gamma^2 < 4k$. Because this is a physical system we know that both γ and k should be positive; clearly this condition also requires that $k > 0$.

b. [5 points] If $x(0) = x_0$, $x'(0) = v_0$, and $\mathcal{L}\{f(t)\} = F(s)$, find the transform $X(s) = \mathcal{L}\{x(t)\}$.

Solution: Applying the Laplace transform to both sides of the differential equation, we have

$$s^2 X - x_0 s - v_0 + \gamma s X - \gamma x_0 + kX = F(s),$$

so that

$$X = \frac{x_0 s + v_0 + \gamma x_0}{s^2 + \gamma s + k} + \frac{F(s)}{s^2 + \gamma s + k}.$$

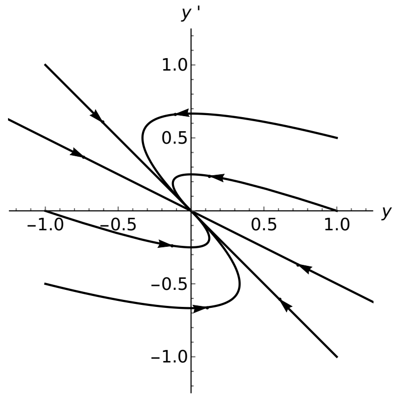
c. [6 points] If $f(t) = 0$, assuming as in (a) that the system is underdamped, invert your transform from (b) to find $x(t)$. *(If you are stuck, assume the equation is $x'' + \gamma x' + \gamma^2 x = 0$.)*

Solution: If the system is underdamped, then we know that the characteristic polynomial $\lambda^2 + \gamma\lambda + k = (\lambda + \frac{\gamma}{2})^2 + k - \frac{\gamma^2}{4}$. Then, with $b = \sqrt{k - \frac{\gamma^2}{4}}$ and $f(t) = 0$, we have

$$\begin{aligned} x = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\{\frac{x_0 s + v_0 + \gamma x_0}{(s + \frac{\gamma}{2})^2 + b^2}\} \\ &= \mathcal{L}^{-1}\{\frac{x_0(s + \frac{\gamma}{2})}{(s + \frac{\gamma}{2})^2 + b^2} + \frac{v_0 + \frac{\gamma}{2}x_0}{(s + \frac{\gamma}{2})^2 + b^2}\} \\ &= x_0 e^{-\gamma t/2}\cos(bt) + \frac{1}{b}(v_0 + \frac{\gamma}{2}x_0)e^{-\gamma t/2}\sin(bt). \end{aligned}$$

For the hint, the characteristic polynomial is $p(s) = s^2 + \gamma s + \gamma^2 = (s + \frac{\gamma}{2})^2 + \frac{3\gamma^2}{4}$, so that we have the answer above with $b = \frac{\sqrt{3}\gamma}{2}$.

7. [12 points] In the following we consider two linear, homogeneous, second-order, constant coefficient differential equations, for $y(t)$ and $z(t)$. The phase portrait for the equation for $y(t)$ is shown to the right, and graphs of $z(t)$ for two different initial conditions are shown in the figure to the right, below. Explain in a sentence or two why each of the following **cannot** be true.



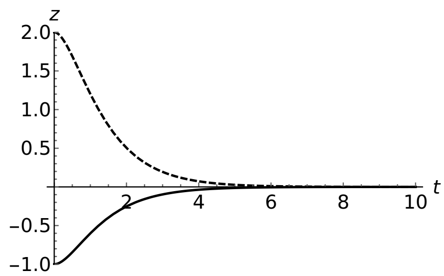
a. [3 points] The equation is $y'' - 3y' + 2y = 0$

Solution: In this case the characteristic equation is $\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$, so that $\lambda = 1$ or $\lambda = 2$, and solutions must grow away from the origin.

b. [3 points] The general solution to the equation is $y = c_1 e^{-t} + c_2 e^{-2t}$.

Solution: Note that if we rewrite the equation as a system, the solution is $\begin{pmatrix} y \\ y' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$, so that the straight line solutions have to be $y = -x$ and $y = -2x$, which is not what we see here (and the direction of fastest collapse is similarly wrong).

c. [3 points] Given some initial conditions, the Laplace transform $Z(s) = \mathcal{L}\{z(t)\} = \frac{2s+4}{s^2+2s+5}$.



Solution: We see from the form of the Laplace transform that the characteristic polynomial is $p(s) = s^2+2s+5 = (s+1)^2+4$, so that roots are $s = -1 \pm 2i$, and solutions should be oscillatory, not exponentially decaying to zero.

d. [3 points] Written as a system, the equation for $z(t)$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Solution: If this were the case we have $\lambda = \pm 2i$, so the general solution to the problem is $z = c_1 \cos(2t) + c_2 \sin(2t)$, which is a pure sinusoid. This is clearly not shown here.