Math 216 — Final Exam

17 December, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Consider the system of differential equations $\mathbf{x}' = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -4 \end{pmatrix} \mathbf{x}$.

a. [6 points] Find the general solution to this system.[1]

Solution: Finding the eigenvalues of the coefficient matrix, we have $\det(\mathbf{A} - \lambda\mathbf{I}) = (-2-\lambda)((1-\lambda)(-4-\lambda)+6) = (-2-\lambda)(\lambda^2+3\lambda+2) = (-2-\lambda)(\lambda+2)(\lambda+1) = 0$, so that $\lambda = -1$ or $\lambda = -2$ (repeated). Then, if $\lambda = -1$, we have for the eigenvector $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. If $\lambda = -2$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that we may take $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Thus the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} e^{-2t}.$$

b. [6 points] Now suppose that we consider only initial conditions in the yz-plane (that is, we take $\mathbf{x}(0) = \begin{pmatrix} 0 \\ y_0 \\ z_0 \end{pmatrix}$). Sketch the phase portrait for these initial conditions, in the yz-plane.

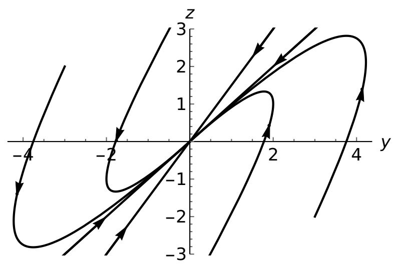
Solution: If we are restricted to the yz-plane, we have only the first and last terms in the general solution,

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} e^{-2t},$$

so that in the yz-plane, we have

$$\begin{pmatrix} y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}.$$

We can sketch the phase portrait in this plane by drawing in the eigenvectors $z = y$ and $z = \frac{3}{2}y$ and the corresponding trajectories, which collaps to the first eigenvector and then to the origin. This is shown below.



[1]Possibly useful: $\det\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}\right) = a(be - cd)$.

2. [12 points] Consider the mass-spring model $2y'' + 4y' + 6y = 8\cos(3t)$.

a. [4 points] Explain why the following statement is true or false: "For any initial condition, the long-term behavior of the mass is the same."

Solution: This is true: we know that the solution to the problem will be $y = y_c + y_p$, where y_c is the solution to the complementary homogeneous problem and y_p a particular solution. All terms in y_c will be exponentially decaying, because the model includes damping. Therefore, as $t \to \infty$, we will end up with $y \to y_p$, for all initial conditions.

b. [4 points] Explain why the following statement is true or false: "The Laplace transform of the steady state solution to the problem is $\mathcal{L}\{y_{ss}\} = \dfrac{8s}{(2s^2+4s+6)(s^2+9)}$."

Solution: This is false, but includes the steady state solution. Note that we know that $y_{ss} = a\cos(3t) + b\sin(3t)$, so that its forward transform will be $\mathcal{L}\{y_{ss}\} = \frac{as+3b}{s^2+9}$, which omits the characteristic polynomial in s.

We can also see this by noting that the indicated expression will decompose by partial fractions as

$$\frac{4s}{(s^2+2s+3)(s^2+9)} = \frac{A(s+1)+B}{(s+1)^2+2} + \frac{Cs+D}{s^2+2},$$

the latter term of which gives the steady state while the first contributes to the response to the initial conditions.

c. [4 points] Explain why the following statement is true or false: "If we change the forcing term to $f(t) = 8\delta(t-7)$, the solution y will have a discontinuity at $t = 7$."

Solution: This is false. The introduction of the delta forcing will result in y having a discontinuous slope at $t = 7$, but the solution itself will be continuous. We can also see this by remembering that the effect of the delta function is to introduce an instantaneous change of one unit in the momentum of the mass, which is proportional to its velocity.

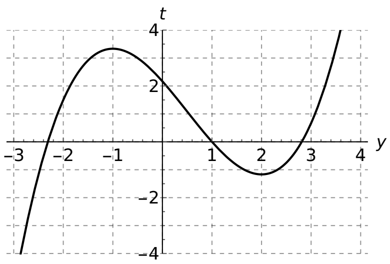
3. [10 points] Consider the initial value problem $y' = (y+1)^{-1}(y-2)^{-1}$, $y(0) = 1$.

a. [4 points] Solve your differential equation to find an implicit solution for y, of the form $t = f(y)$.

Solution: Separating variables, we have $(y^2-y-2)y' = 1$, so that $\frac{1}{3}y^3 - \frac{1}{2}y^2 - 2y = t+C$. Applying the initial condition, $\frac{1}{3} - \frac{1}{2} - 2 = C$, so $C = -\frac{13}{6}$. So

$$t = \frac{1}{6}y(2y^2 - 3y - 12) + \frac{13}{6}.$$

b. [6 points] Suppose that the graph of the $t(y)$ that you found in (a) is shown to the right. Explain what this tells you about the domain (in t) on which the solution to the initial value problem exists, and how that is related to the theory of first-order equations.

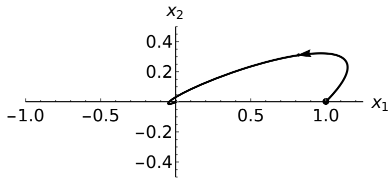


Solution: We note that this gives us the solution to the initial value problem if we reflect it across the line $t = y$. The result will only be a function for $-1 < y < 2$, however; beyond that, the result is not a function. Thus, from the graph, we can see that the domain on which y lives is $-1.167 < t < 3.333$ (ok, from the graph we really can see only that $-1.$(something small) $< t < 3.$(3 or 5); the exact values are $-\frac{7}{6} < t < \frac{10}{3}$). This makes sense, as at those values of t, y takes on the values $y = 2$ and $y = -1$, which are the values at which the equation becomes undefined and thus discontinuous. Our theory indicates that where $y' = f(t, y)$ becomes discontinuous we no longer have any guarantee of a solution, or of it being unique.

4. [12 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a real-valued 2×2 matrix and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

a. [6 points] Suppose that the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda = -1 \pm i$, with $\mathbf{v} = \begin{pmatrix} 2 \pm i \\ 1 \end{pmatrix}$. If $\mathbf{x}$ solves $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, sketch the trajectory for $\mathbf{x}$ in the phase plane.

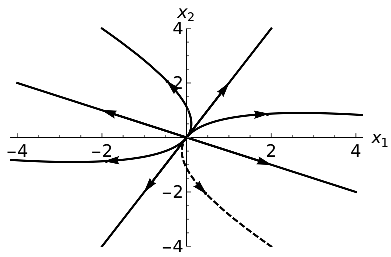
Solution: The general solution is $\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} e^{-t}$, so to satisfy the initial condition we may take $c_1 = 0$ and $c_2 = 1$. Then, as t increases we see that y component of the trajectory increases, and it will be an inward spiral, giving the trajectory shown below.



(If we want to outdo ourselves, we could also note that $x'(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ to give the initial direction of the trajectory, thereby getting the orientation of the spiral shown above.)

b. [6 points] Suppose that eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 2$, with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $\mathbf{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, as $t \to \infty$, which of the following is most correct, and why? (i) $x_2 \approx 2x_1$; (ii) $x_2 \approx -\frac{1}{2}x_1$; (iii) $x_2 \approx -\frac{1}{2}x_1 - 1$; (iv) $x_2 \approx -\frac{1}{2}x_1 - k$, with $k > 1$.

Solution: The phase portrait for the system will look something like the following.



All trajectories end up parallel to the second eigenvector, which has slope $m = -\frac{1}{2}$. Looking at the dashed trajectory, we see that a trajectory through $(0, -1)$, will be shifted significantly below either of $x_2 = -\frac{1}{2}x_1$ or $x_2 = -\frac{1}{2}x_1 - 1$, so (iv) must be correct.

5. [12 points] In lab 6 we considered the Fitzhugh-Nagumo model for the behavior of a neuron,

$$v' = v - \frac{1}{3}v^3 - w + I_{ext}, \qquad \tau w' = v + a - bw.$$

In this problem we analyze this with the parameters $\tau = 1$, $a = \frac{1}{3}$, and $b = 1$.

a. [3 points] Find the v- and w-nullclines, and show that there is a single critical point (v_c, w_c) in this case. Find the critical point in terms of the externally applied voltage I_{ext}.

Solution: The w-nullcline occurs when $w' = 0$, and thus is $w = v + \frac{1}{3}$, The v-nullcline is when $v' = 0$, so when $0 = v - \frac{1}{3}v^3 - w + I_{ext}$. Plugging in for w, we have $0 = -\frac{1}{3}v^3 - \frac{1}{3} + I_{ext}$, so that the critical point is when $v = v_c = \sqrt[3]{3I_{ext} - 1}$ and $w = w_c = \frac{1}{3} + v_c$.

b. [3 points] Linearize the system at the critical point (write your linearization in terms of v_c and w_c—do not plug in the values you found for v_c and w_c). How is the solution to your linearized system related to the solution of the original nonlinear system?

Solution: The Jacobian of the system is $\mathbf{J} = \begin{pmatrix} 1 - v^2 & -1 \\ 1 & -1 \end{pmatrix}$, so $\mathbf{J}(v_c, w_c) = \begin{pmatrix} 1 - v_c^2 & -1 \\ 1 & -1 \end{pmatrix}$ and with $(v, w) = (v_c, w_c) + (x, y)$ the linearization is

$$\mathbf{x}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 - v_c^2 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The solution (x, y) to the linear system tells us the behavior of the nonlinear system when trajectories are sufficiently near the critical point (v_c, w_c). (Note that because the nonlinearity is strictly polynomial, we know that the system is almost linear and the linearization therefore makes sense.)

c. [6 points] Show that the critical point is in this case is always stable. Determine any values of v_c or w_c at which the behavior at the critical point changes. Explain how this result is different from that which you saw in lab.

Solution: The eigenvalues of $\mathbf{J}(v_c, w_c)$ are given by $\det(\mathbf{J}(v_c, w_c) - \lambda\mathbf{I}) = 0$, or $(1 - v_c^2 - \lambda)(-1 - \lambda) + 1 = \lambda^2 + v_c^2\lambda + v_c^2 = 0$. Thus, using the quadratic formula, $\lambda = -\frac{1}{2}v_c^2 \pm \frac{1}{2}\sqrt{v_c^4 - 4v_c^2} = -\frac{1}{2}v_c^2 \pm \frac{1}{2}v_c^2\sqrt{1 - 4v_c^{-2}}$. Because of the square, the leading term is always negative. Then note that $4v_c^{-2} > 0$, so the square root is either real and less than one or complex. Thus the eigenvalues are either both real and negative (if $|v_c| > 2$) or complex with negative real part (if $|v_c| < 2$); at $|v_c| = 2$ the behavior changes from a stable node to a stable spiral. This is different from what we saw in lab, because in that case there was a value of v_c (and hence I_{ext}) at which the critical point went from stable to unstable as well.

Note that the above is predicated on the assumption that $v_c \neq 0$. If $v_c = 0$, we have the degenerate case $\lambda = 0$, twice. This indicates that the linear system is stable (but not asymptotically stable), and in this case we aren't able to speak to the stability of the nonlinear system.

6. [12 points] Consider the nonlinear system

$$x' = 3x - y - x^2, \qquad y' = -\alpha + x - y,$$

where α is a real-valued parameter.

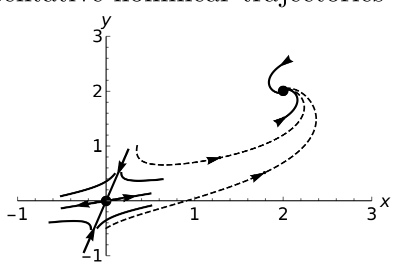
a. [4 points] Find all critical points for the system, and show that for $\alpha > -1$ there are two critical points, if $\alpha = -1$ there is one, and if $\alpha < -1$ there are none.

Solution: From the second equation, at critical points we require $y = x - \alpha$. Plugging into the first, we have $0 = 3x - (x - \alpha) - x^2$, so $x^2 - 2x - \alpha = (x-1)^2 - 1 - \alpha = 0$, and $x = 1 \pm \sqrt{1+\alpha}$. If $\alpha > -1$ there are two critical points, $x = x_{1,2} = 1 \pm \sqrt{1+\alpha}$ (with $y = x_{1,2} - \alpha$); if $\alpha = -1$, there is one, $x_c = 1$ (with $y = 0$); and if $\alpha < -1$ there are no real solutions.

b. [8 points] Let $\alpha = 0$: then the system has two critical points, $(0,0)$ and $(2,2)$. Sketch a phase portrait for the nonlinear system by linearizing at critical points and determining the resulting behavior in the phase plane.

Solution: Note that the Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 3-2x & -1 \\ 1 & -1 \end{pmatrix}$. If $\alpha = 0$, the critical points are $(0,0)$ and $(2,2)$. At $(0,0)$ and $(2,0)$, the Jacobians are, respectively, $\mathbf{J}(0,0) = \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix}$, and $\mathbf{J}(2,2) = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues of the first are given by $(3-\lambda)(-1-\lambda) + 1 = \lambda^2 - 2\lambda - 2 = (\lambda - 1) - 3 = 0$, so $\lambda = 1 \pm \sqrt{3}$. Using the second row of the system, the resulting eigenvectors satisfy $v_1 + (-2 \mp \sqrt{3})v_2 = 0$, so that we may take $\mathbf{v} = \begin{pmatrix} 2 \pm \sqrt{3} \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2 \pm 1.75 \\ 1 \end{pmatrix}$. Thus at $(0,0)$ we have a saddle point with outgoing trajectories along a line with slope approximately $\frac{1}{4}$, and incoming along a line with slope approximately 4.

Similarly, at $(2,2)$ eigenvalues are given by $(\lambda+1)^2 + 1 = 0$, so that $\lambda = -1 \pm i$. At $(1,0)$ we get a slope $(x,y)' = (-1,1)$, so this is an inward counter-clockwise spiral. The resulting phase portrait is shown below, with the trajectories from the linear system as solid curves and some representative nonlinear trajectories with dashed curves.



7. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment.
Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the solution to the initial value problem $W' = \dfrac{-W+5t}{2}$, $W(0) = 6$.

Solution: This is linear, but not separable. Rewriting in standard form, we have $W' + \frac{1}{2}W = \frac{5}{2}t$, so that an integrating factor is $\mu = e^{t/2}$. Multiplying by μ and integrating, we have $e^{t/2}W = \frac{5}{2}\int te^{t/2}\,dt$. Integrating the right-hand side by parts, we have

$$\frac{5}{2}\int te^{t/2}\,dt = 5te^{t/2} - 10e^{t/2} + C,$$

so that $W = 5t - 10 + Ce^{-t/2}$. Applying the initial condition, we have $6 = -10 + C$, so that $C = 16$, and

$$W = 5(t-2) + 16e^{-t/2}.$$

b. [8 points] Find the general solution to the system of first-order linear differential equations, $x' = -8x - y$, $y' = 45x + 4y$.

Solution: In matrix form, this is $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} = \begin{pmatrix} -8 & -1 \\ 45 & 4 \end{pmatrix}$. We look for solutions $\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T = \mathbf{v}e^{\lambda t}$. Then λ and $\mathbf{v}$ are the eigenvalues and eigenvectors of $\mathbf{A}$. Eigenvalues satisfy

$$(-8-\lambda)(4-\lambda) + 45 = \lambda^2 + 4\lambda - 32 + 45 = (\lambda+2)^2 - 4 + 13 = (\lambda+2)^2 + 9 = 0,$$

so that $\lambda = -2 \pm 3i$. Then, if $\lambda = -2+3i$, the eigenvector satisfies $(-6-3i)v_1 - v_2 = 0$, so that we may take $\mathbf{v} = \begin{pmatrix} -1 \\ 6+3i \end{pmatrix}$. Separating the real and imaginary parts of the complex valued solution $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we get the general solution

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -\cos(3t) \\ 6\cos(3t) - 3\sin(3t) \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -\sin(3t) \\ 3\cos(3t) + 6\sin(3t) \end{pmatrix} e^{-2t}.$$

If we instead used the eigenvector $\mathbf{v} = \begin{pmatrix} -6+3i & 45 \end{pmatrix}^T$, we get the general solution $\mathbf{x} = c_1 \begin{pmatrix} -6\cos(3t) - 3\sin(3t) \\ 45\cos(3t) \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3\cos(3t) - 6\sin(3t) \\ 45\sin(3t) \end{pmatrix} e^{-2t}$.

8. [15 points] **DO** complete this problem if you have **NOT** completed the mastery assessment. **DO NOT** complete it if you have completed the mastery assessment.
Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the general solution $Q(t)$ to the differential equation $Q''(t) - 2Q'(t) + 10Q(t) = 30t$.

Solution: We know that the general solution will be $Q = Q_c + Q_p$, where Q_c is the general solution to the complementary homegeneous solution and Q_p is a particular solution. Looking for $Q_c = e^{\lambda t}$, the characteristic equation for λ is $\lambda^2 - 2\lambda + 10 = (\lambda - 1)^2 + 9 = 0$, so that $\lambda = 1 \pm 3i$. Thus $Q_c = c_1 e^t \cos(3t) + c_2 e^t \sin(3t)$. To find Q_p, we guess $Q_p = at + b$, so that (plugging in) $-2a + 10at + 10b = 30t$. Thus $a = 3$ and $-6 + 10b = 0$, so that $b = \frac{3}{5}$. The general solution is thus

$$Q = c_1 e^t \cos(3t) + c_2 e^t \sin(3t) + 3t + \frac{3}{5}.$$

b. [8 points] Find the solution to the inital value problem $v''(t) - 8v'(t) + 25v(t) = 3u_7(t)$, $v(0) = 0$, $v'(0) = 6$.

Solution: Because of the step function on the right-hand side, we choose to use Laplace transforms for this problem. With $V(s) = \mathcal{L}\{v(t)\}$, we have

$$s^2 V - 6 - 8sV + 25V = \frac{3}{s} e^{-7s},$$

so that

$$V = \frac{6}{s^2 - 8s + 25} + \frac{3}{s(s^2 - 8s + 25)} e^{-7s}.$$

Noting that $s^2 - 8s + 25 = (s-4)^2 + 9$, we can invert the first term without further calculation:

$$\mathcal{L}^{-1}\{\frac{6}{(s-4)^2 + 9}\} = 2e^{4t} \sin(3t).$$

Then, for the second term, we use partial fractions, letting $\frac{3}{s(s^2-8s+25)} = \frac{A}{s} + \frac{B(s-4)+C}{(s-4)^2+9}$. Without the exponential, we then have

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{3}{s((s-4)^2+9)}\} &= \mathcal{L}^{-1}\{\frac{A}{s}\} + \mathcal{L}^{-1}\{\frac{B(s-4)}{(s-4)^2+9}\} + \mathcal{L}^{-1}\{\frac{C}{(s-4)^2+9}\} \\
&= A + Be^{4t}\cos(3t) + \frac{1}{3}Ce^{4t}\sin(3t),
\end{aligned}$$

so that

$$v(t) = 2e^{4t}\sin(3t) + \left(A + Be^{4(t-7)}\cos(3(t-7)) + \frac{1}{3}Ce^{4(t-7)}\sin(3(t-7))\right) u_7(t).$$

Finding A, B, and C, we have $A((s-4)^2 + 9) + Bs(s-4) + Cs = 3$, so that with $s = 0$, $A = \frac{3}{25}$; matching terms in s^2, $B = -A = -\frac{3}{25}$; and matching terms in s, $-8A - 4B + C = 0$, so that $C = \frac{12}{25}$.