Math 216
Midterm Exam 1
Fall 2021

Full Name:

uniqname:

ID number:

Instructions:

1. For each problem, put all of your work in the indicated box (if possible, otherwise, indicate clearly in the box where additional details can be found on your paper). To receive credit you must show all of your work for each problem, unless clearly indicated in the problem.
2. The last page of this exam is a table of antiderivatives. You may detach that page if you like, but keep all other pages together, and make sure that any work you want graded is not on the detached page.
3. No calculators, phones, smartwatches. No notecards or notesheets.
4. The *LSA Community Standards of Academic Integrity* are in force, and by taking this exam you agree to be bound by them. DO NOT CHEAT!

1. Choose from among the direction fields below which belongs to each of these ODE (1 point each, no justification needed for this problem):

   (a) $y' = y^2 - 3y + 1$. Direction Field #

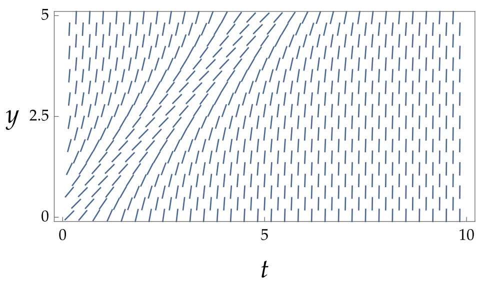
   (b) $y' = y^2 + y + 1$. Direction Field #

   (c) $y' = 1 + (y - t)^2$. Direction Field #

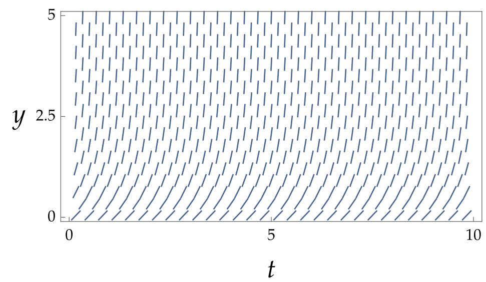
   (d) $y' = 1 - (y - t)^2$. Direction Field #

   (e) $y' = t^2 - 3t + 1$. Direction Field #

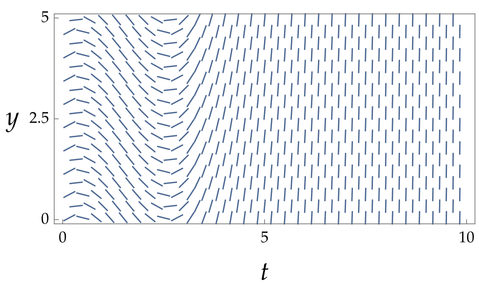
Direction Field # 1



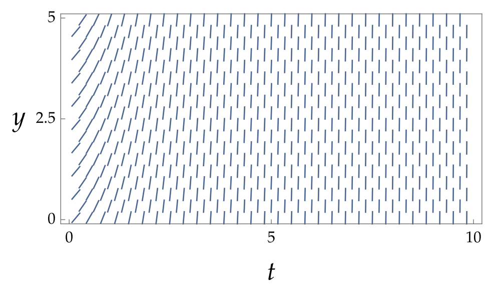
Direction Field # 2



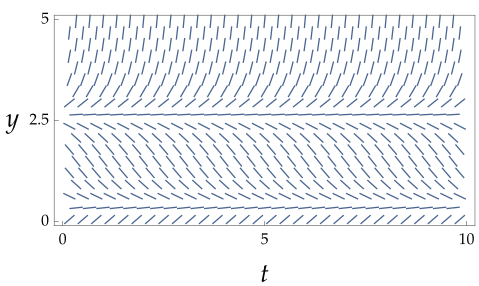
Direction Field # 3



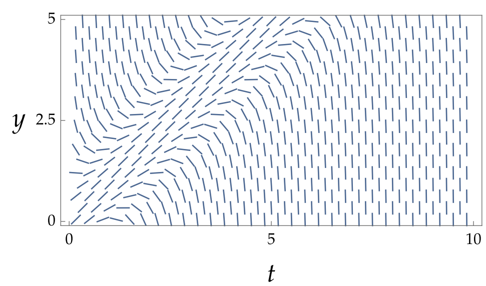
Direction Field # 4



Direction Field # 5



Direction Field # 6



2. (a) (4 points) The temperature T_R of a certain nuclear reactor is decaying exponentially in time toward a background temperature $T_B = 25°$ C: $T_R = T_B + a e^{-t}$ where t is measured in hours and a is a constant. The temperature T of a container of water placed in the reactor is subject to Newton's Law of Cooling/Heating: the time rate of change of the temperature T is proportional to the difference between T and the reactor temperature. If at time $t = 0$,

   - The reactor temperature is at 325° C,
   - The water temperature is the same as the background temperature, and
   - The water temperature is increasing by 600° C per hour,

   find (but do not solve) the precise first-order ODE satisfied by the water temperature $T(t)$.

   (b) (4 points) The number $F(t)$ of fish in a small lake t weeks after the beginning of summer is governed by the differential equation $F' = -4F + S(t)$, where $S(t)$ (fish/week) is the rate at which the lake is stocked with fish by the park ranger. The park ranger starts out putting in 1000 fish/week but gets busy with other things and ends up adding fewer fish every week so that $S(t) = 1000e^{-2t}$. If there aren't any fish in the lake at the beginning of summer, find $F(t)$.

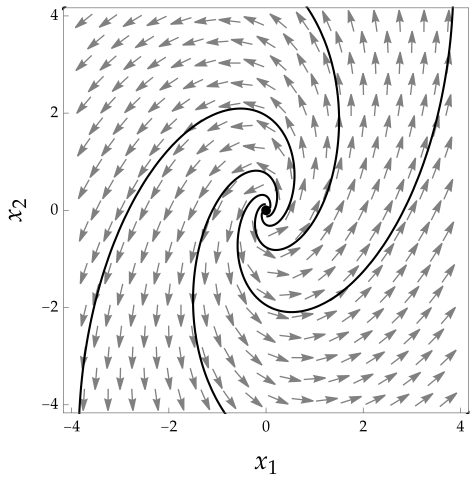
3. Choose from among the given phase portraits the phase portrait for the system $\mathbf{x}' = \begin{pmatrix} 2 & a \\ 4 & 1 \end{pmatrix} \mathbf{x}$ for each given value of a (1 point each, no justification needed for this problem):

   (a) $a = 0$. Phase Portrait # $\boxed{}$

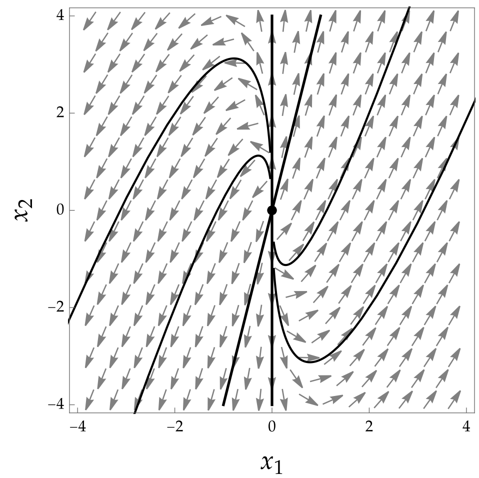
   (b) $a = \frac{15}{16}$. Phase Portrait # $\boxed{}$

   (c) $a = -\frac{26}{16}$. Phase Portrait # $\boxed{}$

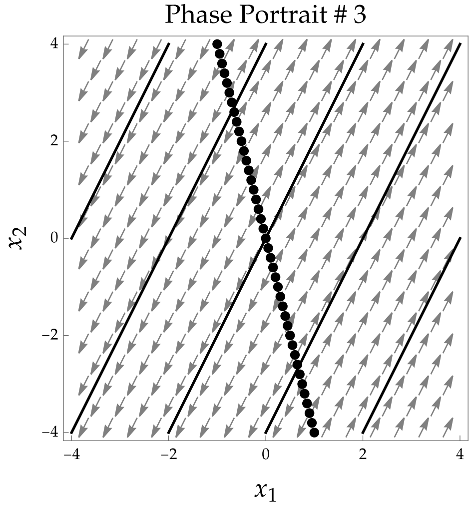
Phase Portrait # 1



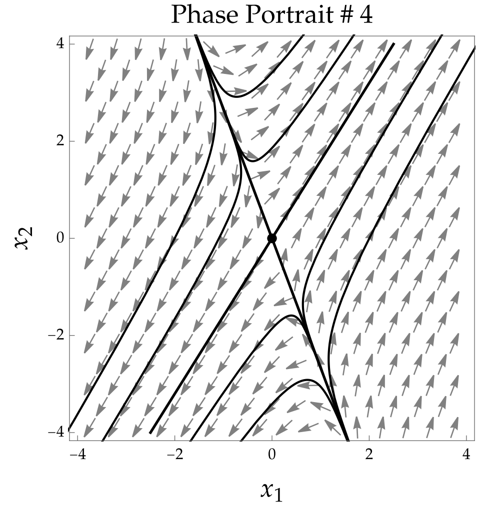
Phase Portrait # 2



Phase Portrait # 3



Phase Portrait # 4



4. (a) (4 points) Solve the initial-value problem $x' = x^2/t + 3x^2t^2$, $x(-1) = \frac{1}{2}$, for $x = x(t)$.

(b) (4 points) A general solution of the differential equation $x' = t/x$ for $x = x(t)$ has the implicit form $x^2 - t^2 = C$. Find the (maximal) interval of existence of the solution with initial condition $x(5) = 4$.

5. (4 points) Consider the initial-value problem for $y(t)$: $y' = \cos(\pi y) + 1$, $y(3) = -2$. Does $\lim_{t\to+\infty} y(t)$ exist? If so, what is it?

6. Consider the system with parameter h

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ -1 & h \end{pmatrix} \mathbf{x}$$

for a vector function $\mathbf{x} = \mathbf{x}(t)$.

(a) (4 points) For which value(s) of h is there a solution of this system of the form

$$\mathbf{x}(t) = \begin{pmatrix} (at+b)\mathrm{e}^{-4t} \\ (t+c)\mathrm{e}^{-4t} \end{pmatrix}$$

for some constants a, b, c? (No need to find a, b, c, just h.)

(b) (4 points) Suppose that $h = -3$. Solve the initial-value problem for the system with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. (a) (4 points) The differential equation $mx'' + \gamma x' + kx = 0$ describes unforced vibrations of an object of mass $m = 1$ kg hanging from a spring with spring constant k kg/s^2 and damped with a dashpot having a damping coefficient γ kg/s. The unknown $x(t)$ is the downward displacement from equilibrium. Suppose that you set the object in motion with initial displacement $x(0) = 0$ m and initial velocity $x'(0) = -6$ m/s (so upwards), and you observe the displacement x at various times t and notice that your data is accurately fit by the curve $x = -2e^{-t}\sin(3t)$. What is the value of the spring constant?

(b) (2 points) Suppose $y_1(x)$ and $y_2(x)$ are both solutions of $y'' - x \cdot y = 0$, and that $W[y_1, y_2](1) = 3$. Find $W[y_1, y_2](x)$ for all x.

8. True or false? Write out the full word "true" or "false" and provide a brief justification (2 points each).

(a) $\mu = 2$ is a bifurcation point for the equation $y' = y^5 + \mu y^4 + y^3$.

(b) There is exactly one differentiable function $y(t)$ defined near $t = 1$ whose graph passes through the point $(1, 2)$ and such that $y(t)y'(t)$ and $1 + y(t)\sin(t)$ are actually the same functions of t.

(c) There is a matrix $\mathbf{A}$ with the following eigenvalues/eigenvectors:

$$\lambda_1 = 2,\ \mathbf{x}_1 = \begin{pmatrix} -20 \\ 25 \end{pmatrix}; \quad \lambda_2 = 65,\ \mathbf{x}_2 = \begin{pmatrix} 40 \\ -50 \end{pmatrix}.$$

(d) There is some continuous function $f(y)$ for which $y' = f(y)$ has only two equilibria, both unstable.

$f(x)$	Anti-derivative $F(x)$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin\ x$	$-\cos\ x + C$
$\cos\ x$	$\sin\ x + C$
$\sec^2 x$	$\tan\ x + C$
$\tan\ x\ \sec\ x$	$\sec\ x + C$
a (constant)	$ax + C$
$\frac{1}{a^2 + x^2}$ (a is a constant)	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{a^2 - x^2}}$ (a is a constant)	$\arcsin\left(\frac{x}{a}\right) + C$
$\sinh\ x$	$\cosh\ x + C$
$\cosh\ x$	$\sinh\ x + C$
e^x	$e^x + C$

$f(x)$	Anti-derivative $F(x)$
$\tan\ x$	$-\ln\lvert\cos\ x\rvert + C$
$\cot\ x$	$\ln\lvert\sin\ x\rvert + C$
$\sec\ x$	$\ln\lvert\sec\ x + \tan\ x\rvert + C$
$\csc\ x$	$\ln\lvert\csc\ x - \cot\ x\rvert + C$