Math 216
Midterm Exam 2
Fall 2021

Full Name:

uniqname:

ID number:

Instructions:

1. For each problem, put all of your work in the indicated box (if possible, otherwise, indicate clearly in the box where additional details can be found on your paper). To receive credit you must show all of your work for each problem, unless clearly indicated in the problem. Work on scrap paper will not be graded, so make sure all of your important work and answers are written on the test paper.
2. No calculators, phones, smartwatches. No notecards or notesheets.
3. The *LSA Community Standards of Academic Integrity* are in force, and by taking this exam you agree to be bound by them. DO NOT CHEAT!

1. Consider an "RLC" circuit in which the capacitor charge $Q(t)$ satisfies $LQ'' + RQ' + C^{-1}Q = E(t)$ where L, R, C are the inductance (Henries), resistance (Ohms), and capacitance (Farads), and where $E(t)$ is a variable source of voltage (Volts). Suppose that $R = 2$ Ohms and $C = 1/5$ Farads, and that the voltage source is sinusoidal: $E(t) = \cos(5t)$.

   (a) (3 Points.) Find the steady-state periodic response, keeping the inductance L as a variable parameter in your answer.

   (b) (2 Points.) Find the amplitude of the steady-state periodic response from part (a), and then determine the value of inductance L that maximizes it.

2. (5 Points.) For certain initial conditions, the displacement $x(t)$ of a mass from equilibrium in a mechanical system without any damping or forcing is given by

$$x(t) = -\sqrt{3}\cos(4\pi t) + \sin(4\pi t).$$

Write $x(t)$ in phase/amplitude form and use your answer to find the *second* positive time $t > 0$ at which the mass passes equilibrium. Note that for some angles in the first quadrant we have

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

3. In a lab there are five identical "RLC" circuits in each of which the capacitor charge satisfies $LQ'' + RQ' + C^{-1}Q = E(t)$ with $L = 3$ Henries, $R = 4$ Ohms, $C = 0.5$ Farads. The five circuits are driven by five time-dependent voltage sources $E(t)$ and have different amounts $Q(0)$ of initial charge on the capacitor and different amounts $Q'(0)$ of current flowing at time zero, according to this table:

Circuit	Capacitor charge	Voltage source $E(t)$	$Q(0)$	$Q'(0)$
1	$Q = Q_1(t)$	$E_1(t) = 1/(1+t^2)$ Volts	0.25 Coul.	0 Amp.
2	$Q = Q_2(t)$	$E_2(t) = 1/(1+t^2)$ Volts	0 Coul.	1 Amp.
3	$Q = Q_3(t)$	$E_3(t) \equiv 6$ Volts	0.25 Coul.	0 Amp.
4	$Q = Q_4(t)$	$E_4(t) = (25+24t^2)/(1+t^2)$ Volts	1 Coul.	1 Amp.
5	$Q = Q_5(t)$	For $E_5(t)$, see part (d)	0 Coul.	0 Amp.

(a) (3 Points.) Express $Q_4(t)$ in terms of $Q_1(t)$, $Q_2(t)$, and/or $Q_3(t)$.

(b) (2 Points.) Find $\lim_{t\to+\infty}[Q_1(t) - Q_2(t)]$.

(c) (2 Points.) Would the answer to part (b) be different if $R = 0$ Ohms instead? Why or why not?

(d) (3 Points.) Circuit number 5 has a "pulsed" voltage source $E(t)$ that is zero except on the time interval $1 < t < 6$ seconds, at the beginning of which it is suddenly switched on to 10 Volts, and during which it increases exponentially following $E(t) = 10e^{b(t-1)}$ for some rate b sec^{-1}. If instantaneously after switching on, $E'(1) = 30$ Volts/sec, use the definition of the Laplace transform to find $\mathscr{L}\{E(t)\}$ (but don't solve for $Q_5(t)$).

4. (5 Points.) Using the method of undetermined coefficients, solve the initial-value problem $y'' - 4y = 12e^{-2t}$, $y(0) = 0$, $y'(0) = 1$ for $y(t)$.

5. (6 Points.) The differential equation $t^2y'' - 2ty' + 2y = 0$ has the following two solutions: $y = t$ and $y = t^2$. Assuming that $t > 0$, solve the initial-value problem $t^2y'' - 2ty' + 2y = t^2$, $y(1) = 1$, $y'(1) = 0$.

6. (4 Points.) Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and assume that $x(t)$ and $y(t)$ satisfy the initial conditions $x(0) = 0$ and $y(0) = 1$. Let $g(t)$ be a function having a Laplace transform denoted $G(s)$, for s large enough. Find $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$ in terms of $G(s)$. Your answers should be in terms of s.

7. (5 Points.) In solving an initial-value problem for a certain linear second-order ODE by Laplace transforms, we arrive at

$$Y(s) = \frac{s}{s^3 - 3s^2 + 9s - 27} = \frac{s}{(s-3)(s^2+9)}.$$

What is $y(t)$?

8. True or false? Write out the full word "true" or "false" and provide a brief justification (2 points each).

(a) Variation of parameters is applicable to find a solution of $y'' + \sin(y) = \tan(t)$.

(b) The function $f(t) = \mathrm{e}^{\sqrt{t}}$ has a Laplace transform.

(c) There is a piecewise-continuous function of exponential order having Laplace transform $F(s) = \dfrac{s-1}{s+1}$ for $s > -1$.

(d) An unforced mechanical system described by the ODE $2y'' + y' + 2y = 0$ is underdamped.

(e) A sinusoidally-forced mechanical system described by the ODE $2y'' + y' + 2y = F_0 \cos(\Omega t)$ has a steady state sinusoidal solution with an amplitude that can be an arbitrarily large multiple of $|F_0|$ if Ω is chosen appropriately.