**Math 216**
**Midterm Exam 1**
**Fall 2021**

Full Name: SOLUTIONS

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Instructions:

1. For each problem, put all of your work in the indicated box (if possible, otherwise, indicate clearly in the box where additional details can be found on your paper). To receive credit you must show all of your work for each problem, unless clearly indicated in the problem.
2. The last page of this exam is a table of antiderivatives. You may detach that page if you like, but keep all other pages together, and make sure that any work you want graded is not on the detached page.
3. No calculators, phones, smartwatches. No notecards or notesheets.
4. The *LSA Community Standards of Academic Integrity* are in force, and by taking this exam you agree to be bound by them. DO NOT CHEAT!

1. Choose from among the direction fields below which belongs to each of these ODE (1 point each, no justification needed for this problem):

   (a) $y' = y^2 - 3y + 1$. Direction Field # $\boxed{5}$
   Autonomous, so the segment slopes can't depend on $t$. Also there are two roots of $y^2 - 3y + 1$ so equilibrium solutions should be present.

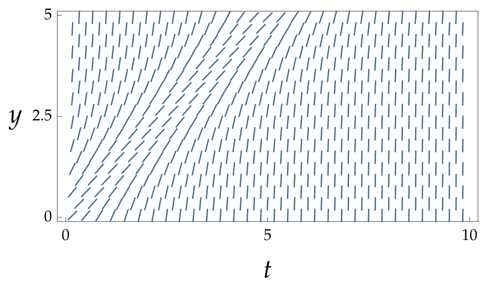
   (b) $y' = y^2 + y + 1$. Direction Field # $\boxed{2}$
   Autonomous again, but now there are no (real) roots of $y^2 + y + 1$ so no equilibria.

   (c) $y' = 1 + (y - t)^2$. Direction Field # $\boxed{1}$
   Slope depends on both $y$ and $t$, but is always $\geq 1$, so no horizontal tangents.

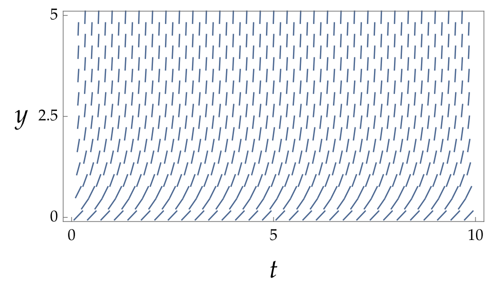
   (d) $y' = 1 - (y - t)^2$. Direction Field # $\boxed{6}$
   Slope depends on both $y$ and $t$, but horizontal tangents occur at points where $y = t \pm 1$.

   (e) $y' = t^2 - 3t + 1$. Direction Field # $\boxed{3}$
   Slope depends on $t$ only; should see horizontal tangents along two vertical lines (roots of $t^2 - 3t + 1$).

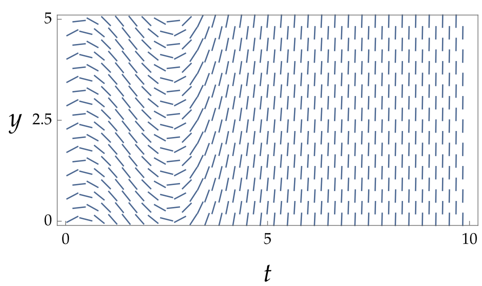
Direction Field # 1



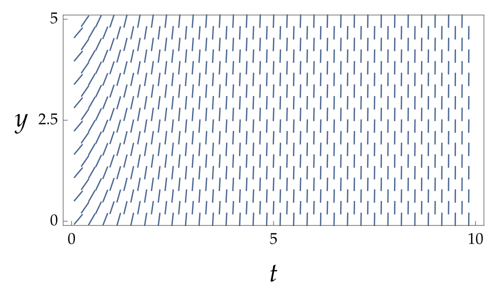

Direction Field # 2

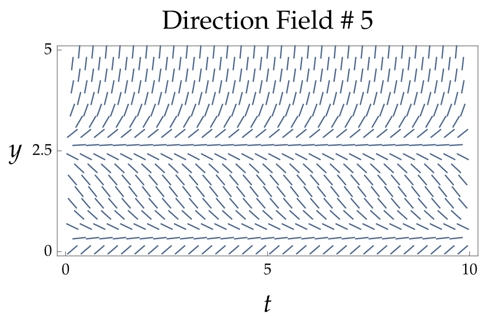



Direction Field # 3

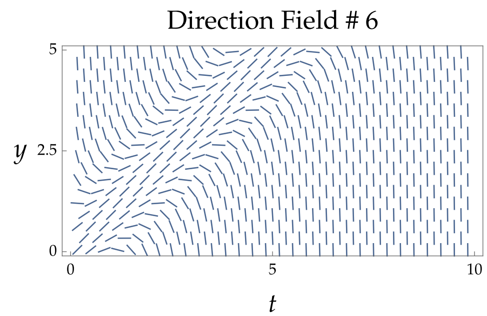



Direction Field # 4





Direction Field # 5
y
5
2.5
0
0
5
10
t

Direction Field # 6
y
5
2.5
0
0
5
10
t

2. (a) (4 points) The temperature $T_R$ of a certain nuclear reactor is decaying exponentially in time toward a background temperature $T_B = 25^\circ$ C: $T_R = T_B + a\mathrm{e}^{-t}$ where $t$ is measured in hours and $a$ is a constant. The temperature $T$ of a container of water placed in the reactor is subject to Newton's Law of Cooling/Heating: the time rate of change of the temperature $T$ is proportional to the difference between $T$ and the reactor temperature. If at time $t = 0$,

- The reactor temperature is at $325^\circ$ C,
- The water temperature is the same as the background temperature, and
- The water temperature is increasing by $600^\circ$ C per hour,

find (but do not solve) the precise first-order ODE satisfied by the water temperature $T(t)$.

Solution: To have $T_R = 325^\circ$ C at $t = 0$ requires that $a = 300^\circ$ C, so the time-dependent reactor temperature is $T_R = 25 + 300\mathrm{e}^{-t}$. Newton's Law of Cooling/Heating says that there is some real constant $k$ of proportionality so that

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - T_R(t)) = k(T - 25 - 300\mathrm{e}^{-t}).$$

Putting $t = 0$, $T = 25^\circ$ C, and $\mathrm{d}T/\mathrm{d}t = 600^\circ$ C per hour then gives $600 = -300k$ so $k = -2$. Therefore the differential equation for $T(t)$ is

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}t} = -2(T - 25 - 300\mathrm{e}^{-t}).}$$

(b) (4 points) The number $F(t)$ of fish in a small lake $t$ weeks after the beginning of summer is governed by the differential equation $F' = -4F + S(t)$, where $S(t)$ (fish/week) is the rate at which the lake is stocked with fish by the park ranger. The park ranger starts out putting in 1000 fish/week but gets busy with other things and ends up adding fewer fish every week so that $S(t) = 1000\mathrm{e}^{-2t}$. If there aren't any fish in the lake at the beginning of summer, find $F(t)$.

Solution: An integrating factor is $\mathrm{e}^{4t}$, so the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{4t}F(t)) = 1000\mathrm{e}^{2t} \implies \mathrm{e}^{4t}F(t) = 500\mathrm{e}^{2t} + C$$

where $C$ is a constant of integration. Putting $F(0) = 0$ and setting $t = 0$ gives $C = -500$.

$$\boxed{F(t) = 500\mathrm{e}^{-2t} - 500\mathrm{e}^{-4t}.}$$

An alternate approach that is equally effective and straightforward is to use the variation of parameters method from Written HW 1.

3. Choose from among the given phase portraits the phase portrait for the system $\mathbf{x}' = \begin{pmatrix} 2 & a \\ 4 & 1 \end{pmatrix} \mathbf{x}$ for each given value of $a$ (1 point each, no justification needed for this problem):

(a) $a = 0$. Phase Portrait # $\boxed{2}$

The eigenvalues in general are $\lambda = \frac{1}{2}(3 \pm \sqrt{16a+1})$. Taking $a = 0$ gives $\lambda = 1, 2$ so we have an unstable node.

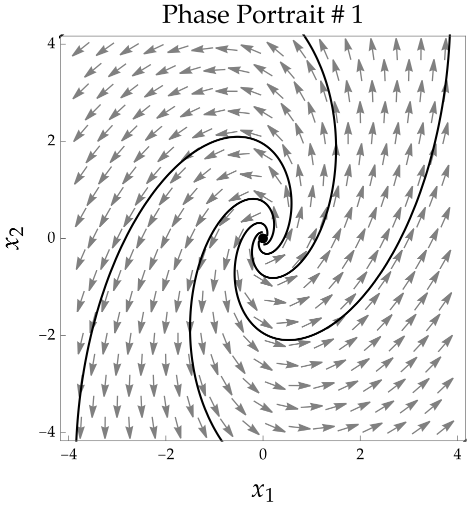
(b) $a = \frac{15}{16}$. Phase Portrait # $\boxed{4}$

For $a = \frac{15}{16}$ the eigenvalues are $\lambda = -\frac{1}{2}, \frac{7}{2}$, so we have a saddle point.

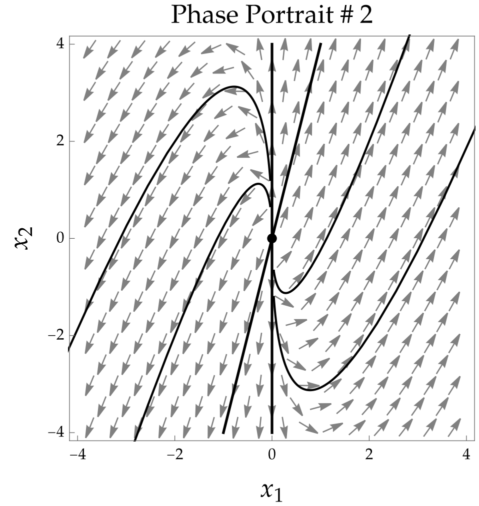
(c) $a = -\frac{26}{16}$. Phase Portrait # $\boxed{1}$

Taking $a = -\frac{26}{16}$ the eigenvalues are $\lambda = \frac{1}{2}(3 \pm 5\mathrm{i})$, so we have an unstable spiral point.

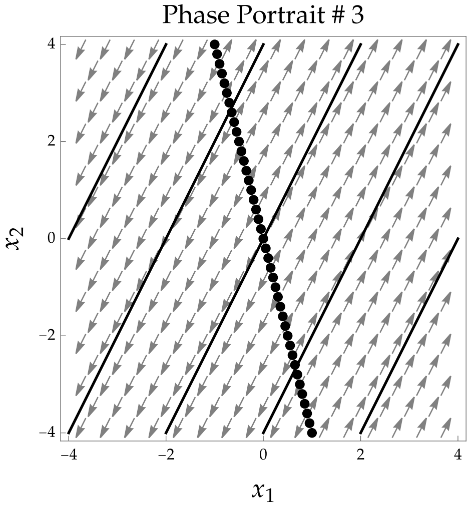
Phase Portrait # 1



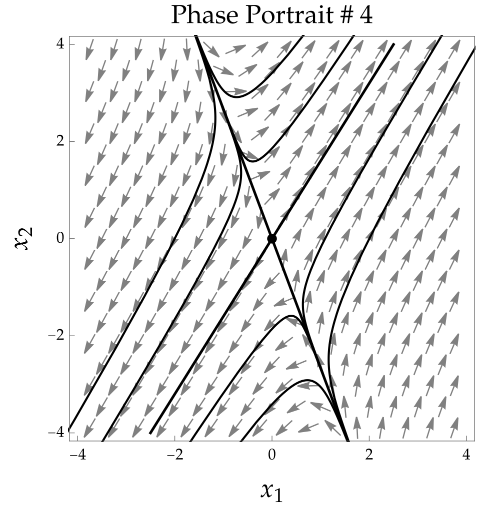

Phase Portrait # 2




Phase Portrait # 3




Phase Portrait # 4



4. (a) (4 points) Solve the initial-value problem $x' = x^2/t + 3x^2t^2$, $x(-1) = \frac{1}{2}$, for $x = x(t)$.

Solution: separating the variables gives $x^{-2}\,\mathrm{d}x = (t^{-1} + 3t^2)\,\mathrm{d}t$ so $-x^{-1} = \ln(|t|) + t^3 + C$. Putting in $(t, x) = (-1, \frac{1}{2})$ gives $C = -1$. So

$$\boxed{x(t) = \frac{1}{1 - t^3 - \ln(|t|)} = \frac{1}{1 - t^3 - \ln(-t)}.}$$

(b) (4 points) A general solution of the differential equation $x' = t/x$ for $x = x(t)$ has the implicit form $x^2 - t^2 = C$. Find the (maximal) interval of existence of the solution with initial condition $x(5) = 4$.

Solution: putting $t = t_0 = 5$ and $x = x_0 = 4$ gives $C = -9$. Taking square roots, there are then two possible solutions for this value of $C$, namely $x(t) = \pm\sqrt{t^2 - 9}$. To have $x(5) = 4$ then requires picking the positive sign so $x(t) = \sqrt{t^2 - 9}$. This is real and differentiable for $t < -3$ and for $t > 3$, only the latter of which is an interval containing $t_0 = 5$. So the interval of existence is

$$\boxed{(3, +\infty).}$$

5. (4 points) Consider the initial-value problem for $y(t)$: $y' = \cos(\pi y) + 1$, $y(3) = -2$. Does $\lim_{t\to+\infty} y(t)$ exist? If so, what is it?

Solution: the equilibria are the roots of $\cos(\pi y) + 1$ which are all of the odd integers. Since $\cos(\pi y) + 1 \geq 0$, each of these roots is a semistable equilibrium, attractive from the left and repulsive on the right. So by the phase line method and the fact that $y(3) = -2$ lies halfway between two equilibria,

$$\boxed{\lim_{t\to+\infty} y(t) = -1.}$$

6. Consider the system with parameter $h$

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ -1 & h \end{pmatrix} \mathbf{x}$$

for a vector function $\mathbf{x} = \mathbf{x}(t)$.

(a) (4 points) For which value(s) of $h$ is there a solution of this system of the form

$$\mathbf{x}(t) = \begin{pmatrix} (at+b)\mathrm{e}^{-4t} \\ (t+c)\mathrm{e}^{-4t} \end{pmatrix}$$

for some constants $a, b, c$? (No need to find $a, b, c$, just $h$.)

Solution: We need that the coefficient matrix has a repeated eigenvalue $\lambda = -4$. The characteristic equation for the problem reads $(h-\lambda)(-3-\lambda)+1=0$ or $\lambda^2+(3-h)\lambda+1-3h=0$. The discriminant is $(3-h)^2-4+12h = h^2+6h+5 = (h+5)(h+1)$. So we have a repeated root if either $h=-5$ or $h=-1$. In those cases, the repeated root is $\lambda = (h-3)/2$ which is $-4$ only for $h=-5$ (it is $-2$ if $h=-1$ instead). Therefore the only possible value of $h$ is

$$\boxed{h = -5.}$$

Another approach would be to first insist that $\lambda = -4$ is an eigenvalue. Since the characteristic equation is linear in $h$, this automatically and immediately yields $h=-5$. However, one should then confirm that this eigenvalue is repeated when $h=-5$ because otherwise the solution does not include any term proportional to $t\mathrm{e}^{-4t}$. A third approach is to deduce two conditions (for $\lambda = -4$ to be an eigenvalue and for it to be repeated) by simply equating the characteristic polynomial to $(\lambda+4)^2$, and then check that all the coefficients of powers of $\lambda$ match if and only if $h=-5$.

(b) (4 points) Suppose that $h=-3$. Solve the initial-value problem for the system with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution: The characteristic equation reads $\lambda^2+6\lambda+10=0$ which has complex-conjugate roots $\lambda = -3 \pm \mathrm{i}$. An eigenvector for $\lambda = -3+\mathrm{i}$ is $\mathbf{v} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$, so the general solution is $c_1\mathbf{u}(t) + c_2\mathbf{w}(t)$, where $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are the real and imaginary parts of the vector solution $\mathrm{e}^{-3t}\mathrm{e}^{\mathrm{i}t}\mathbf{v} = \mathrm{e}^{-3t}(\cos(t)+\mathrm{i}\sin(t))\mathbf{v}$. Therefore $\mathbf{u}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$ and $\mathbf{w}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$. At $t=0$ the general solution reads $c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so we need $c_1 = c_2 = 1$. The solution is therefore

$$\boxed{\mathbf{x}(t) = \mathrm{e}^{-3t}\begin{pmatrix} \cos(t)+\sin(t) \\ \cos(t)-\sin(t) \end{pmatrix}.}$$

7. (a) (4 points) The differential equation $mx'' + \gamma x' + kx = 0$ describes unforced vibrations of an object of mass $m = 1$ kg hanging from a spring with spring constant $k$ kg/s$^2$ and damped with a dashpot having a damping coefficient $\gamma$ kg/s. The unknown $x(t)$ is the downward displacement from equilibrium. Suppose that you set the object in motion with initial displacement $x(0) = 0$ m and initial velocity $x'(0) = -6$ m/s (so upwards), and you observe the displacement $x$ at various times $t$ and notice that your data is accurately fit by the curve $x = -2\mathrm{e}^{-t}\sin(3t)$. What is the value of the spring constant?

   Solution: The solution is a linear combination of $\mathrm{e}^{-t}\cos(3t)$ and $\mathrm{e}^{-t}\sin(3t)$ so the roots of the characteristic equation have to be $-1 \pm 3\mathrm{i}$. Since the characteristic equation here is $\lambda^2 + \gamma\lambda + k = 0$ which has roots $(-\gamma \pm \sqrt{\gamma^2 - 4k})/2$, we need to have $\gamma = 2$ and then $4k - \gamma^2 = 36$ or

$$\boxed{k = 40/4 = 10 \text{ kg/s}^2.}$$

   Another method of obtaining this result that takes much more effort and is more prone to computational errors would be to simply plug the given solution into the ODE and require that it is true as an identity. Then one has to separate terms proportional to $\sin(3t)$ and $\cos(3t)$ to get two equations on $\gamma$ and $k$ and solve them. Of course it gives the same result.

   (b) (2 points) Suppose $y_1(x)$ and $y_2(x)$ are both solutions of $y'' - x \cdot y = 0$, and that $W[y_1, y_2](1) = 3$. Find $W[y_1, y_2](x)$ for all $x$.

   Solution: since this linear and homogeneous second order ODE when written in the general form $y'' + p(x)y' + q(x)y = 0$ has $p(x) \equiv 0$, Abel's Theorem says that Wronskians of any pair of solutions must be independent of $x$. So

$$\boxed{W[y_1, y_2](x) \equiv 3.}$$

8. True or false? Write out the full word "true" or "false" and provide a brief justification (2 points each).

   (a) $\mu = 2$ is a bifurcation point for the equation $y' = y^5 + \mu y^4 + y^3$.
   True. Since $y^5 + \mu y^4 + y^3 = y^3(y^2 + \mu y + 1)$, the critical points are $y = 0$ and $y = \frac{1}{2}(-\mu \pm \sqrt{\mu^2 - 4})$ provided the latter two are real. So there is just one equilibrium if $-2 < \mu < 2$ and three equilibria if $\mu < -2$ or $\mu > 2$.

   (b) There is exactly one differentiable function $y(t)$ defined near $t = 1$ whose graph passes through the point $(1, 2)$ and such that $y(t)y'(t)$ and $1 + y(t)\sin(t)$ are actually the same functions of $t$.
   True. This is the same question as whether there exists a unique solution of the initial-value problem $y' = f(t, y)$ where $f(t, y) := y^{-1} + \sin(t)$ with initial condition $y(1) = 2$. Since $f_y(t, y) = -y^{-2}$, we see that $f(t, y)$ and $f_y(t, y)$ are continuous on the whole $(t, y)$-plane except where $y = 0$. The point $(1, 2)$ is not on the horizontal line $y = 0$, so we can center it within a rectangle like $0 < t < 2$ and $1 < y < 3$ on which both functions are continuous. So by Theorem 2.4.2 there is a unique solution.

   (c) There is a matrix $\mathbf{A}$ with the following eigenvalues/eigenvectors:

$$\lambda_1 = 2,\ \mathbf{x}_1 = \begin{pmatrix} -20 \\ 25 \end{pmatrix}; \quad \lambda_2 = 65,\ \mathbf{x}_2 = \begin{pmatrix} 40 \\ -50 \end{pmatrix}.$$

   False. This matrix has distinct eigenvalues, but the given eigenvectors are proportional, which is not possible according to Theorem 3.1.3.

   (d) There is some continuous function $f(y)$ for which $y' = f(y)$ has only two equilibria, both unstable.
   False. The function $f(y)$ would have to have just two roots, and be positive to the right of each and negative to the left of each. Since there is an interval between the two roots, the function would have to be both positive and negative on this interval.

| $f(x)$ | Anti-derivative $F(x)$ |
|---|---|
| $x^n,\ n \neq -1$ | $\dfrac{x^{n+1}}{n+1} + C$ |
| $\dfrac{1}{x}$ | $\ln\lvert x\rvert + C$ |
| $\sin\ x$ | $-\cos\ x + C$ |
| $\cos\ x$ | $\sin\ x + C$ |
| $\sec^2 x$ | $\tan\ x + C$ |
| $\tan\ x\ \sec\ x$ | $\sec\ x + C$ |
| $a$ (constant) | $ax + C$ |
| $\dfrac{1}{a^2 + x^2}$ ($a$ is a constant) | $\dfrac{1}{a}\ \arctan\left(\dfrac{x}{a}\right) + C$ |
| $\dfrac{1}{\sqrt{a^2 - x^2}}$ ($a$ is a constant) | $\arcsin\left(\dfrac{x}{a}\right) + C$ |
| $\sinh\ x$ | $\cosh\ x + C$ |
| $\cosh\ x$ | $\sinh\ x + C$ |
| $e^x$ | $e^x + C$ |

| $f(x)$ | Anti-derivative $F(x)$ |
|---|---|
| $\tan\ x$ | $-\ln\lvert\cos\ x\rvert + C$ |
| $\cot\ x$ | $\ln\lvert\sin\ x\rvert + C$ |
| $\sec\ x$ | $\ln\lvert\sec\ x + \tan\ x\rvert + C$ |
| $\csc\ x$ | $\ln\lvert\csc\ x - \cot\ x\rvert + C$ |