Math 216
Midterm Exam 2
Fall 2021

Full Name: SOLUTIONS

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Instructions:

1. For each problem, put all of your work in the indicated box (if possible, otherwise, indicate clearly in the box where additional details can be found on your paper). To receive credit you must show all of your work for each problem, unless clearly indicated in the problem. Work on scrap paper will not be graded, so make sure all of your important work and answers are written on the test paper.
2. No calculators, phones, smartwatches. No notecards or notesheets.
3. The *LSA Community Standards of Academic Integrity* are in force, and by taking this exam you agree to be bound by them. DO NOT CHEAT!

1. Consider an "RLC" circuit in which the capacitor charge $Q(t)$ satisfies $LQ'' + RQ' + C^{-1}Q = E(t)$ where L, R, C are the inductance (Henries), resistance (Ohms), and capacitance (Farads), and where $E(t)$ is a variable source of voltage (Volts). Suppose that $R = 2$ Ohms and $C = 1/5$ Farads, and that the voltage source is sinusoidal: $E(t) = \cos(5t)$.

   (a) (3 Points.) Find the steady-state periodic response, keeping the inductance L as a variable parameter in your answer.

   Solution: to find the steady-state periodic response we use the method of undetermined coefficients: $Q(t) = A\cos(5t) + B\sin(5t)$. Then $Q'(t) = -5A\sin(5t) + 5B\cos(5t)$ and $Q''(t) = -25A\cos(5t) - 25B\cos(5t)$, so substitution into $LQ'' + 2Q' + 5Q = \cos(5t)$ gives

$$\begin{aligned} (5-25L)A + 10B &= 1 \\ -10A + (5-25L)B &= 0 \end{aligned}$$

   where the first line comes from the terms proportional to $\cos(5t)$ and the second line comes from the terms proportional to $\sin(5t)$. Solving for A and B gives

$$A = \frac{5-25L}{(5-25L)^2+100} \quad \text{and} \quad B = \frac{10}{(5-25L)^2+100}.$$

   The steady-state periodic response is therefore

$$\boxed{Q(t) = \frac{5-25L}{(5-25L)^2+100}\cos(5t) + \frac{10}{(5-25L)^2+100}\sin(5t).}$$

   (b) (2 Points.) Find the amplitude of the steady-state periodic response from part (a), and then determine the value of inductance L that maximizes it.

   Solution: the steady-state response amplitude is $R = \sqrt{A^2+B^2}$ or

$$R = \sqrt{\frac{(5-25L)^2+100}{[(5-25L)^2+100]^2}} = \frac{1}{\sqrt{(5-25L)^2+100}}.$$

   Maximizing this is the same as minimizing the quantity under the radical, which occurs when $5 - 25L = 0$ or

$$\boxed{L = \tfrac{1}{5} = 0.2 \text{ Henries.}}$$

2. (5 Points.) For certain initial conditions, the displacement $x(t)$ of a mass from equilibrium in a mechanical system without any damping or forcing is given by

$$x(t) = -\sqrt{3}\cos(4\pi t) + \sin(4\pi t).$$

Write $x(t)$ in phase/amplitude form and use your answer to find the *second* positive time $t > 0$ at which the mass passes equilibrium. Note that for some angles in the first quadrant we have

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Solution: the amplitude is

$$R = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2,$$

and the phase satisfies $\cos(\delta) = -\sqrt{3}/2$ and $\sin(\delta) = 1/2$. So δ is an angle in the second quadrant. Since $\arccos(\cdot)$ returns an angle in the first or second quadrant, we can take $\delta = \arccos(-\sqrt{3}/2)$. Equivalently, we can relate δ to an angle θ in the first quadrant by $\delta + \theta = \pi$, for which $\cos(\theta) = \cos(\pi - \delta) = -\cos(\delta) = \sqrt{3}/2$ and $\sin(\theta) = \sin(\pi - \delta) = \sin(\delta) = 1/2$, so using the given function values, $\theta = \pi/6$. Hence $\delta = \pi - \pi/6 = 5\pi/6$. So, the phase/amplitude form is

$$\boxed{x(t) = 2\cos(4\pi t - 5\pi/6).}$$

We have $x(t) = 0$ whenever $4\pi t - 5\pi/6$ is an odd multiple of $\pi/2$, i.e., whenever $8t - 5/3$ is an odd integer, i.e., whenever $8t = 5/3$ plus an odd integer. The first positive t corresponds to taking -1 for the odd integer, and the second positive t corresponds to taking 1 for the odd integer. So, the time we seek satisfies $8t = 5/3 + 3/3 = 8/3$ or

$$\boxed{t = \tfrac{1}{3}.}$$

3. In a lab there are five identical "RLC" circuits in each of which the capacitor charge satisfies $LQ'' + RQ' + C^{-1}Q = E(t)$ with $L = 3$ Henries, $R = 4$ Ohms, $C = 0.5$ Farads. The five circuits are driven by five time-dependent voltage sources $E(t)$ and have different amounts $Q(0)$ of initial charge on the capacitor and different amounts $Q'(0)$ of current flowing at time zero, according to this table:

Circuit	Capacitor charge	Voltage source $E(t)$	$Q(0)$	$Q'(0)$
1	$Q = Q_1(t)$	$E_1(t) = 1/(1+t^2)$ Volts	0.25 Coul.	0 Amp.
2	$Q = Q_2(t)$	$E_2(t) = 1/(1+t^2)$ Volts	0 Coul.	1 Amp.
3	$Q = Q_3(t)$	$E_3(t) \equiv 6$ Volts	0.25 Coul.	0 Amp.
4	$Q = Q_4(t)$	$E_4(t) = (25+24t^2)/(1+t^2)$ Volts	1 Coul.	1 Amp.
5	$Q = Q_5(t)$	For $E_5(t)$, see part (d)	0 Coul.	0 Amp.

(a) (3 Points.) Express $Q_4(t)$ in terms of $Q_1(t)$, $Q_2(t)$, and/or $Q_3(t)$.

Solution: we apply the superposition principle for nonhomogeneous linear ODE, noticing that each equation has exactly the same linear operator $LD^2 + RD + C^{-1}$ on the left-hand side. So, we write $Q_4(t) = aQ_1(t) + bQ_2(t) + cQ_3(t)$ and try to determine a, b, c. The initial condition $Q_4(0) = 1$ then reads $\frac{1}{4}a + \frac{1}{4}c = 1$, and $Q_4'(0) = 1$ reads $b = 1$. To fully determine a, c, we require that the differential equation for $Q_4(t)$ holds, which means that

$$\frac{25+24t^2}{1+t^2} = E_4(t) = aE_1(t) + bE_2(t) + cE_3(t) = \frac{a}{1+t^2} + \frac{1}{1+t^2} + 6c.$$

The numerator over the common denominator of $1+t^2$ gives us the equation $24(1+t^2) = a + 6c(1+t^2)$, so from the coefficients of t^2 we get that $c = 4$. Taking the constant terms, or going back to $Q_4(0) = 1$, one finds that $a = 0$. So $a = 0$, $b = 1$, and $c = 4$, and therefore by superposition,

$$\boxed{Q_4(t) = Q_2(t) + 4Q_3(t).}$$

(b) (2 Points.) Find $\lim_{t\to+\infty}[Q_1(t) - Q_2(t)]$.

Solution: The difference $\Delta Q(t) := Q_1(t) - Q_2(t)$ is a solution of the associated homogeneous equation $L\Delta Q'' + R\Delta Q' + C^{-1}\Delta Q = 0$. Since the coefficients are all positive, $\Delta Q(t)$ is a transient response that decays to zero as $t \to +\infty$. It is not necessary to know what the initial conditions are, nor is it necessary to use the fact that the forcing function is $E(t) = 1/(1+t^2)$. We conclude that

$$\boxed{\lim_{t\to+\infty}[Q_1(t) - Q_2(t)] = 0.}$$

(c) (2 Points.) Would the answer to part (b) be different if $R = 0$ Ohms instead? Why or why not?

Solution: yes it would be different, because then the difference $Q_1(t) - Q_2(t)$ satisfies $L\Delta Q'' + C^{-1}\Delta Q = 0$, the general solution of which is simple harmonic motion with natural frequency of vibration $\Omega_0 = 1/\sqrt{LC}$; this does not decay to zero as $t \to +\infty$, so in general the limit of $\Delta Q(t)$ as $t \to +\infty$ does not exist.

(d) (3 Points.) Circuit number 5 has a "pulsed" voltage source $E(t)$ that is zero except on the time interval $1 < t < 6$ seconds, at the beginning of which it is suddenly switched on to 10 Volts, and during which it increases exponentially following $E(t) = 10e^{b(t-1)}$ for some rate b sec^{-1}. If instantaneously after switching on, $E'(1) = 30$ Volts/sec, use the definition of the Laplace transform to find $\mathscr{L}\{E(t)\}$ (but don't solve for $Q_5(t)$).

Solution: over the time interval $1 < t < 6$, we have $E(t) = 10e^{b(t-1)}$, and since this implies that $E'(1) = 10b$, we need $b = 3$ sec^{-1} to match a rate of change of 30 Volts/sec. Then, using the definition,

$$\mathscr{L}\{E(t)\} = \int_1^6 E(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_1^6 10\mathrm{e}^{3(t-1)}\mathrm{e}^{-st}\,\mathrm{d}t = 10\mathrm{e}^{-3}\int_1^6 \mathrm{e}^{(3-s)t}\,\mathrm{d}t = 10\mathrm{e}^{-3}\frac{\mathrm{e}^{(3-s)6} - \mathrm{e}^{(3-s)1}}{3-s}.$$

4. (5 Points.) Using the method of undetermined coefficients, solve the initial-value problem $y'' - 4y = 12e^{-2t}$, $y(0) = 0$, $y'(0) = 1$ for $y(t)$.

Solution: first we solve the associated homogeneous problem: $y'' - 4y = 0$. The characteristic equation reads $\lambda^2 - 4 = 0$ so $\lambda = 2, -2$. The complementary function is therefore $y_c(t) = c_1 e^{2t} + c_2 e^{-2t}$. Next we find a particular solution using the method of undetermined coefficients. Because e^{-2t} is a solution of the associated homogeneous problem, we assume the form $y = Y(t) = Ate^{-2t}$. Then, some computation gives:

$$Y'(t) = Ae^{-2t} - 2Ate^{-2t},$$

$$Y''(t) = -4Ae^{-2t} + 4Ate^{-2t},$$

so substitution gives $[-4Ae^{-2t} + 4Ate^{-2t}] - 4[Ate^{-2t}] = 12e^{-2t}$ so $A = -3$. The general solution is therefore

$$y = y_c(t) + Y(t) = -3te^{-2t} + c_1 e^{2t} + c_2 e^{-2t}.$$

Likewise,

$$y' = -3e^{-2t} + 6te^{-2t} + 2c_1 e^{2t} - 2c_2 e^{-2t}.$$

Therefore, $y(0) = c_1 + c_2$ and $y'(0) = -3 + 2c_1 - 2c_2$. Imposing the initial conditions then gives $c_1 = 1$ and $c_2 = -1$. The solution is

$$\boxed{y(t) = -3te^{-2t} + e^{2t} - e^{-2t}.}$$

5. (6 Points.) The differential equation $t^2y'' - 2ty' + 2y = 0$ has the following two solutions: $y = t$ and $y = t^2$. Assuming that $t > 0$, solve the initial-value problem $t^2y'' - 2ty' + 2y = t^2$, $y(1) = 1$, $y'(1) = 0$.

Solution: we know that the general solution is $y(t) = y_c(t) + Y(t)$ where $y_c(t) = c_1t + c_2t^2$ is the complementary function (because t and t^2 are linearly independent, in fact $W[t,t^2](t) = t^2$ which is nonzero on the interval $(0, +\infty)$ containing the initial point $t = 1$). To find a particular solution, we need to use the method of variation of parameters. Putting the problem in standard form (divide by t^2), it reads $y'' - 2t^{-1}y' + 2t^{-2}y = g(t) = 1$. Looking for a particular solution in the form $Y(t) = u_1(t)t + u_2(t)t^2$, we solve the system

$$\begin{aligned} tu_1'(t) + t^2u_2'(t) &= 0 \\ u_1'(t) + 2tu_2'(t) &= g(t) = 1 \end{aligned}$$

for $u_1'(t)$ and $u_2'(t)$. We get $u_1'(t) = -1$ and $u_2'(t) = t^{-1}$. Any antiderivatives will do to obtain a particular solution, so we take $u_1(t) = -t$ and $u_2(t) = \ln(|t|)$. So the particular solution we get is $Y(t) = -t^2 + t^2\ln(|t|)$. The general solution is therefore

$$y(t) = -t^2 + t^2\ln(|t|) + c_1t + c_2t^2,$$

and taking a derivative,

$$y'(t) = -2t + t + 2t\ln(|t|) + c_1 + 2c_2t.$$

Therefore, $y(1) = -1 + c_1 + c_2$ and $y'(1) = -1 + c_1 + 2c_2$. Imposing the initial conditions then gives $c_1 = 3$ and $c_2 = -1$, so

$$\boxed{\begin{aligned} y(t) &= -t^2 + t^2\ln(|t|) + 3t - t^2 \\ &= -2t^2 + t^2\ln(|t|) + 3t. \end{aligned}}$$

In all of these formulas, we could have written $\ln(t)$ instead of $\ln(|t|)$ because $t > 0$.

6. (4 Points.) Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} g(t) \\ 0 \end{pmatrix}, \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

and assume that $x(t)$ and $y(t)$ satisfy the initial conditions $x(0) = 0$ and $y(0) = 1$. Let $g(t)$ be a function having a Laplace transform denoted $G(s)$, for s large enough. Find $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$ in terms of $G(s)$. Your answers should be in terms of s.

The equations of the system read:

$$\begin{aligned} x' &= x + 2y + g(t) \\ y' &= 2x + y \end{aligned}$$

Taking the Laplace transform of both equations using the initial conditions:

$$\begin{aligned} sX(s) &= X(s) + 2Y(s) + G(s) \\ sY(s) - 1 &= 2X(s) + Y(s) \end{aligned}$$

We rearrange this as a linear system for $X(s)$ and $Y(s)$:

$$\begin{pmatrix} 1-s & 2 \\ 2 & 1-s \end{pmatrix} \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} -G(s) \\ -1 \end{pmatrix}.$$

Taking the inverse matrix (using the fact that its determinant is $s^2 - 2s - 3 = (s-3)(s+1)$) we have

$$\begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \frac{1}{(s-3)(s+1)} \begin{pmatrix} 1-s & -2 \\ -2 & 1-s \end{pmatrix} \begin{pmatrix} -G(s) \\ -1 \end{pmatrix}.$$

So,

$$\boxed{X(s) = \frac{(s-1)G(s) + 2}{(s-3)(s+1)}, \quad Y(s) = \frac{2G(s) + s - 1}{(s-3)(s+1)}.}$$

7. (5 Points.) In solving an initial-value problem for a certain linear second-order ODE by Laplace transforms, we arrive at

$$Y(s) = \frac{s}{s^3 - 3s^2 + 9s - 27} = \frac{s}{(s-3)(s^2+9)}.$$

What is $y(t)$?

Solution: We set up the partial-fraction expansion

$$Y(s) = \frac{s}{s^3 - 3s^2 + 9s - 27} = \frac{A}{s-3} + \frac{Bs+C}{s^2+9}.$$

Multiplying through by the common denominator gives $s = A(s^2+9) + (Bs+C)(s-3)$. Setting $s = 3$ gives $A = \frac{1}{6}$. Then setting $s = 0$ gives $0 = \frac{9}{6} - 3C$ so $C = \frac{1}{2}$. Matching the coefficients of the terms proportional to s gives $1 = -3B + C = -3B + \frac{1}{2}$ so $B = -\frac{1}{6}$. Therefore,

$$Y(s) = \frac{1}{6}\frac{1}{s-3} - \frac{1}{6}\frac{s}{s^2+9} + \frac{1}{2}\frac{1}{s^2+9}.$$

Multiplying and dividing the last term by 3 then allows us to use the table of transforms to find

$$y(t) = \frac{1}{6}e^{3t} - \frac{1}{6}\cos(3t) + \frac{1}{6}\sin(3t).$$

8. True or false? Write out the full word "true" or "false" and provide a brief justification (2 points each).

   (a) Variation of parameters is applicable to find a solution of $y'' + \sin(y) = \tan(t)$.
   False. This is a nonlinear problem.

   (b) The function $f(t) = e^{\sqrt{t}}$ has a Laplace transform.
   True. $e^{\sqrt{t}}$ is continuous for $t \geq 0$, and the inequality $\sqrt{t} \leq t$ holds for all $t \geq 1$, so also $e^{\sqrt{t}} \leq e^{t}$ holds for all $t \geq 1$. Therefore the transform exists at least for $s > 1$ (actually, for $s > 0$).

   (c) There is a piecewise-continuous function of exponential order having Laplace transform $F(s) = \dfrac{s-1}{s+1}$ for $s > -1$.
   False. The Laplace transform of every piecewise-continuous function of exponential order tends to zero as $s \to +\infty$, but $F(s) \to 1$ as $s \to +\infty$ instead. Another way to see this is to write $F(s)$ in the form $F(s) = 1 - 2/(s+1)$. Then from the table, $f(t) = \delta(t) - 2e^{-t}$, but $\delta(t)$ is not a piecewise-continuous function (it is not even a function, just a generalized function).

   (d) An unforced mechanical system described by the ODE $2y'' + y' + 2y = 0$ is underdamped.
   True. The discriminant of the characteristic equation $2\lambda^2 + \lambda + 2 = 0$ is $1^2 - 4 \cdot 2 \cdot 2 = -15 < 0$. Therefore we have distinct complex-conjugate roots and by definition the system is underdamped.

   (e) A sinusoidally-forced mechanical system described by the ODE $2y'' + y' + 2y = F_0 \cos(\Omega t)$ has a steady state sinusoidal solution with an amplitude that can be an arbitrarily large multiple of $|F_0|$ if Ω is chosen appropriately.
   False. This system has positive damping $\gamma = 1$, so the gain has a finite maximum value.