Math 216 — First Midterm

13 February, 2013

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

This material is (c)2016, University of Michigan Department of Mathematics, and released under a Creative Commons By-NC-SA 4.0 International License. It is explicitly not for distribution on websites that share course materials.

1. [14 points] Find explicit general, real-valued solutions for each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $\frac{dy}{dx} = -\frac{\cos x}{\sin x}\, y + \frac{1}{\sin x}$.

b. [7 points] $\frac{dy}{dx} - (x-1)y^2 = x - 1$.

2. [14 points] Solve each of the following to find explicit real-valued solutions for y. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $y'' + 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 4$.

b. [7 points] $y''' + 4y'' = 0$, $y(0) = 0$, $y'(0) = \pi$, $y''(0) = e$.

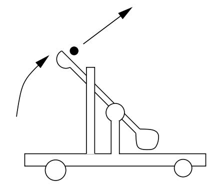
3. [16 points] Consider the differential equation $ty'' + 2y' = 0$ on the domain $0 < t < \infty$.

a. [6 points] Show that $y_1 = 1$, $y_2 = \frac{1}{t}$ and $y_3 = \frac{t-3}{t}$ are all solutions to this differential equation.

b. [5 points] Are y_1, y_2 and y_3 linearly independent? Explain.

c. [5 points] Write the general solution to this differential equation.

4. [10 points] Suppose we launch a 16 lb bowling ball from a catapult, as suggested in the figure to the right. In this problem we consider the vertical velocity v of the bowling ball. We shall assume that the initial vertical velocity is 45 ft/s, and that the bowling ball is released from a height of 50 ft. Gravity provides a downward acceleration of 32 ft/s^2, and the force of air resistance is proportional to the square of the velocity with constant of proportionality $k = 0.0005$. With these assumptions, the bowling ball reaches its apogee (highest point) of $h = 80.7$ ft at $t = 1.38$ seconds.



a. [6 points] Write an initial value problem for the vertical velocity of the bowling ball on its ascent. Note that you do not need to solve this problem.

b. [4 points] Write an initial value problem for the vertical velocity of the bowling ball on its descent. Note that you do not need to solve this problem.

5. [16 points] Consider a differential equation $y' = f(x, y)$ with initial condition $y(0) = 1$. Using two different numerical methods, we obtain the following approximations to the solution of this initial value problem. Note that the error in the approximations is included in the tables.

Method 1:

x	0	0.5	1.0	1.5	2.0
y	1	1	1.1980	1.4238	1.5949
error	0	0.1071	0.1408	0.0794	−0.0358

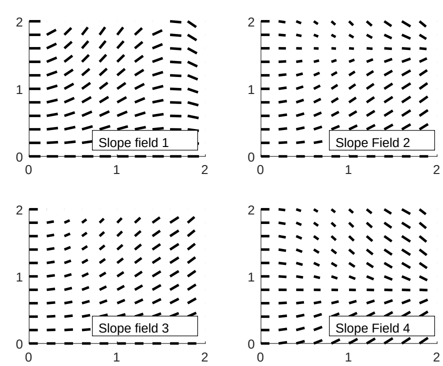
Method 2:

x	0	0.5	1.0	1.5	2.0
y	1	1.1137	1.3365	1.4558	1.4854
error	0	−0.0066	0.0023	0.0475	0.0736

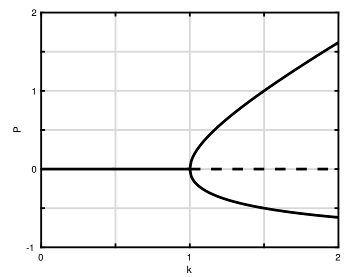
a. [3 points] What is the value of h used in the numerical approximations?

b. [7 points] One of the methods shown is Euler's method, and the other is improved Euler. Which is which? Why?

c. [6 points] Given the data above, which of the slope fields to the right could be the slope field for this differential equation? Explain.



6. [16 points] Consider a animal population modeled by a differential equation $P' = f(P)$, where the function $f(P)$ involves a parameter k. At $k = 1$ there is a bifurcation point, as shown in the bifurcation diagram to the right. In this figure, solid curves indicate stable solutions while dashed curves indicate unstable ones. Even though $P < 0$ is not physically realizable, include negative values of P in your analysis in parts (a) and (b) below.

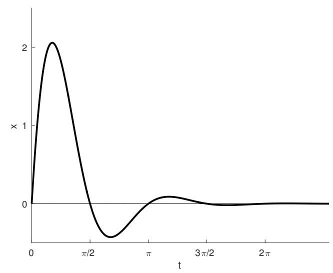


a. [6 points] Sketch phase diagrams for the differential equation $P' = f(P)$ for $k = 0.5$, $k = 1$ and $k = 1.5$.

b. [6 points] Sketch qualitatively reasonable solution curves this equation for the case $k = 1.5$.

c. [4 points] Thinking of P as an animal population, what is the implication of the bifurcation point? Give a possible explanation for what k could measure.

7. [14 points] Consider a spring that, when suspended vertically, is stretched 2 meters by a mass of 2 kg. (For this problem take g, the acceleration due to gravity, to be $g = 10$ m/s^2.) When the mass is attached to the spring and it is positioned horizontally so that the mass can slide back and forth, the motion of the mass is given by



$$2x'' + cx' + kx = 0.$$

a. [3 points] Use the information about the vertical stretch of the spring to find the value of the constant k.

b. [7 points] If the mass is set in motion from equilibrium with an initial velocity $v(0) = 8$ m/s, the resulting motion is shown in the figure to the right, above. If we want the motion of the mass to not have any oscillatory characteristics, should we increase or decrease the value of c? If we do this, for what value of c will there first be no oscillation?

c. [4 points] What value of c gives the motion shown in the figure above?