Math 216 — Second Midterm

28 March, 2013

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] For this problem note that the general solution to $y'' + 5y' + 4y = 0$ is $y = c_1e^{-t} + c_2e^{-4t}$. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find a real-valued general solution to

$$y'' + 5y' + 4y = 3e^{-4t}.$$

b. [8 points] Find the solution to the

$$y'' + 5y' + 4y = 16t, \qquad y(0) = 2, \quad y'(0) = -2.$$

2. [12 points] The eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ -5 & 7 \end{pmatrix}$ are $\lambda = 4 \pm 4i$. Use the eigenvalue method to find a real-valued general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. *(Note that minimal partial credit will be given on this problem.)*

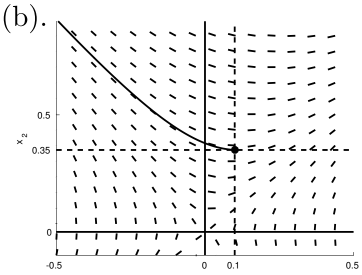
3. [16 points] Consider the system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1$$

a. [8 points] Find a real-valued general solution to this system.

b. [4 points] Find the particular solution if $x_1(0) = 0.10$, $x_2(0) = 0.35$.

c. [4 points] Briefly explain why the direction field and solution trajectory shown to the right could not match this system and your solution from (b).

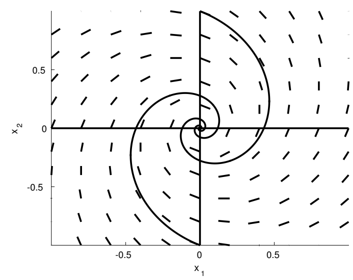


4. [12 points] Three linear constant-coefficient homogeneous systems are described below. Included in the description is one of the following three charateristics; for each, specify the missing two characteristics by circling the correct answers.

1. whether it is a node, saddle point or spiral point, and
2. whether the equilibrium point $(0,0)$ is asymptotically stable, stable or unstable;
3. the sign of the constant a in the system.

No explanation is necessary for your answers.

a. [4 points] The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ having direction field and representative trajectories in the phase plane shown in the figure to the right, if $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ a & 1 \end{pmatrix}$ and $a > 0$.



This is a

node saddle point spiral point

and is

asymptotically stable stable unstable

b. [4 points] The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if the equilibrium point is a saddle point and the eigenvalues are $\lambda = 3$ and $\lambda = a$.

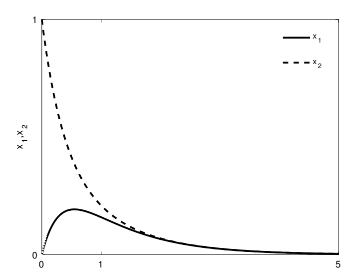
The equilibrium point is

asymptotically stable stable unstable

and a is

< 0 > 0

c. [4 points] The system $x_1' = -2x_1 + ax_2$, $x_2' = x_1 - 2x_2$ whose solution with the initial conditions $x_1(0) = 0$, $x_2(0) = 1$ is shown in the figure to the right, if the equilibrium point is asymptotically stable.



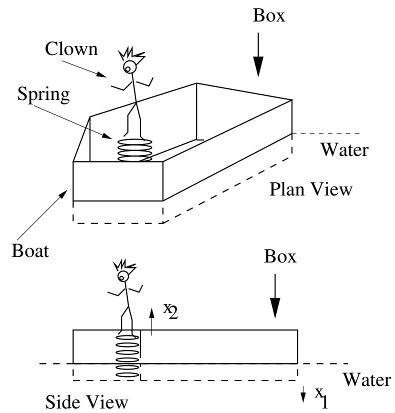
The equilibrium point is a

node saddle point spiral point

and a is

< 0 > 0

5. [16 points] Consider a clown on a spring in a boat, as suggested by the figure to the right. At time $t = 0$ we place a large box in the boat. Then, with some not entirely unreasonable assumptions, the displacement x_1 of the boat and x_2 of the clown are given by



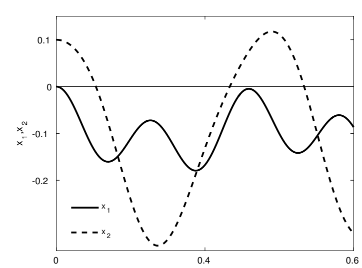
$$\begin{aligned} x_1'' &= -425x_1 + 75x_2 - 35 \\ x_2'' &= 150x_1 - 150x_2. \end{aligned}$$

(Here, x_1 and x_2 are measured in meters and t in seconds.) Letting $\mathbf{A} = \begin{pmatrix} -425 & 75 \\ 150 & -150 \end{pmatrix}$, the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda = -600$ and $\lambda = -125$ with $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

a. [6 points] What are the natural frequencies at which the boat and clown will oscillate? Explain.

b. [6 points] Find the general solution to the homogeneous system associated with this system.

c. [4 points] A solution to this system is shown to the right. What initial conditions were applied to x_1 and x_2 to obtain this solution?



6. [15 points] Consider a mass-spring system modeled by

$$y'' + cy' + ky = f(t),$$

where c is the damping coefficient associated with the system and k the spring constant. For each of the following give a short explanation of your answers.

a. [5 points] If $f(t) = 3\sin(2t)$ and the system is at resonance, are c and k positive, negative or zero? Give specific values for c and k if possible.

b. [5 points] If $f(t) = 3\sin(2t)$ and the system is at resonance, sketch a qualitatively accurate graph of y_p, the particular solution to the problem.

c. [5 points] If $c > 0$, $k > 0$, and $f(t) = 3\sin(2t)$, what can you say (without solving the differential equation) about the long-term behavior of y?

7. [14 points] In this problem we consider the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & t/2 \\ -4t^{-3} & t^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with the initial condition $\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ $(a, b \neq 0)$.

a. [4 points] Find the Euler's method approximation for the solution to the system after one step with a step size h (your answer will involve a, b and h). What is the meaning of your result?

b. [4 points] Rewrite your approximation in the form $\mathbf{P}\begin{pmatrix} a \\ b \end{pmatrix}$. What is $\mathbf{P}$?

c. [2 points] Find $\det(\mathbf{P})$.

d. [4 points] Is it possible that the Euler step could end at $x_1 = 0$, $x_2 = 0$? Explain.