Math 216 — Final Exam

26 April, 2013

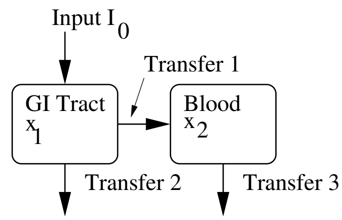
This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Solve each of the following, as indicated.
Note that little partial credit will be given in this problem.

a. [6 points] Find an explicit solution to $x^2\, y\, y' = y^2 + 1$, $y(1) = 1$.

b. [6 points] Find a general real-valued solution to $y''' - y'' + 4y' - 4y = 3e^t$ if one solution to the associated homogeneous equation is $y = e^t$.

2. [12 points] Consider the two compartment system shown in the figure to the right. We suppose that it models a drug taken orally, so that the input I_0 enters the gastrointestinal tract, resulting in an amount x_1 of the drug there. This is transferred to the blood and removed at rates proportional to the amount present, as suggested by the figure. The amount of the drug in the blood is then x_2, and this is reduced at a rate proportional to the amount present as well.



a. [2 points] Explain why this results in the system of equations

$$x_1' = -(k_1 + k_2)x_1 + I_0$$
$$x_2' = k_1 x_1 - k_3 x_2$$

b. [4 points] Suppose that with some appropriate assumptions $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, and $I_0 = 12$. Take $x_1(0) = x_2(0) = 0$ and solve the system by finding x_1 and using this to find x_2.

Problem 2, continued. We are considering the system

$$x_1' = -4x_1 + 12, \qquad x_2' = x_1 - 2x_2.$$

c. [3 points] This system can be written as a linear second-order constant-coefficient equation for x_2. Based on your solution in (b), what is the characteristic equation for this second order equation?

d. [3 points] Finally, write the system as a 2×2 system of equations in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

3. [12 points] Consider a grand stage show put on to celebrate the conclusion of your math 216 experience: hundreds of magicians crowd a stage, and every minute,

- 20% of the magicians leave the stage;
- each magician produces—*from thin air*—one rabbit and 4 flowers;
- each rabbit eats 2% of the flowers on the stage, and 10% of the rabbits hop off-stage in search of other food.

With x_1 = the number of magicians on stage, x_2 = the number of rabbits, and x_3 = the number of flowers, this leads to the system of equations

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1/5 & 0 & 0 \\ 1 & -1/10 & 0 \\ 4 & 0 & -1/50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Find the general real-valued solution to this problem using the eigenvalue method.

4. [12 points] Find each of the indicated Laplace or inverse Laplace transforms. Note that if the problem indicates that you should use the definition of the transform, looking up the solution in the table at the end of the exam will be worth zero points.

a. [4 points] Find $\mathfrak{L}^{-1}\{\frac{2}{(s^2+9)(s+2)}\}$

b. [4 points] Find $\mathfrak{L}^{-1}\{\frac{s}{(s^2+4)^2}\}$

c. [4 points] Use the definition of the Laplace transform to show that $\mathfrak{L}\{f'(t)\} = sF(s) - f(0)$, where $\mathfrak{L}\{f(t)\} = F(s)$.

5. [14 points] Solve using Laplace transforms:

$$y'' + 2y' + y = e^{-t}(1 - u(t-2)), \qquad y(0) = 3, \quad y'(0) = 4.$$

6. [12 points] For each of the following, circle True or False to indicate whether the statement is true or not, and provide a one-sentence explanation. ***Note*** *that without an explanation no credit will be awarded.*

a. [3 points] Given any two solutions y_1 and y_2 to an equation $y'' + p(x)y' + q(x)y = 0$ (where $p(x)$ and $q(x)$ are continuous), any other solution to the equation may be written as $y = c_1y_1 + c_2y_2$ for some constants c_1 and c_2.

True False

b. [3 points] $\mathfrak{L}^{-1}\{\frac{3s-6}{s^2+4s+20}\} = 3e^{-2t}\cos(4t) - \frac{3}{2}e^{-2t}\sin(4t)$.

True False

c. [3 points] Any 2nd or higher order system of ordinary differential equations $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$ $(n \geq 2)$ may be written as a system of first-order ordinary differential equations.

True False

d. [3 points] Suppose we approximate the solution to $y' = f(x, y)$ on the domain $0 \leq x \leq 10$ using a numerical method. With $h = .1$ we get $y(10) \approx 1.501$; with $h = .01$, $y(10) \approx 1.487$; and with $h = 0.001$ or $h = 0.0001$, $y(10) \approx 1.486$. Thus the exact value of $y(10)$ is to three decimal places 1.486, the errors when $h = 0.1$ and $h = 0.01$ are 0.015 and 0.001 respectively, and it is most likely that the numerical method used was the improved Euler method.

True False

7. [14 points] The van der Pohl oscillator is a circuit that may be modeled with the system of differential equations

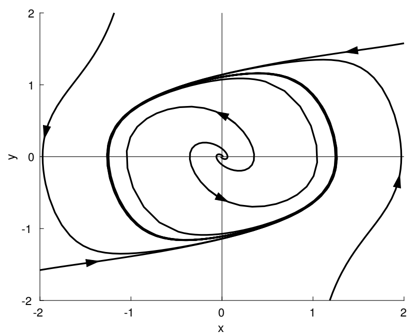
$$x' = -y, \qquad y' = x + (a - y^2)\,y,$$

where x is the charge on a capacitor in the circuit and y is current in the circuit, scaled appropriately. The constant a is a parameter in the system.

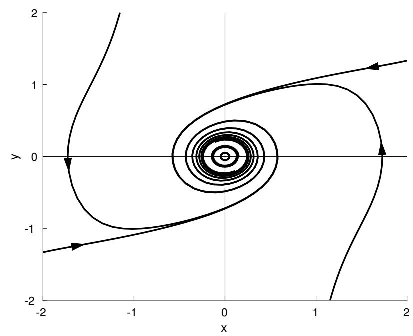
a. [3 points] Find all critical points for this system.

b. [6 points] The two phase portraits (I and II) shown below are generated for the system two of the three cases $a = -1$, $a = 0$ or $a = 1$. By doing a linear analysis of the system at your critical points, determine which cases these match and explain why.

I.



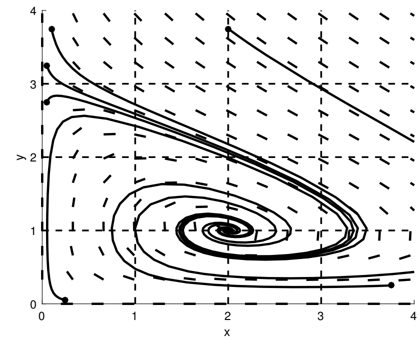
II.



c. [5 points] Based on your linear analysis, sketch a phase portrait for the last of the three cases $a = -1$, $a = 0$, or $a = 1$.

8. [12 points] Consider the system

$$x' = ax + 2xy$$
$$y' = 3y - y^2 + bxy$$



where a and b are constants. The direction field and phase portrait for the system are shown in the figure to the right. (Dots indicate initial conditions for the trajectories shown.)

a. [6 points] What are a and b? Be sure to explain your answer.

b. [6 points] Suppose that x and y are populations of interacting species. What type of interaction is being modeled here? Explain what the phase portrait shown tells you about the behavior and expected long-term values of the populations and sketch a representative solution (x and y) against t.