Math 216 — First Midterm

13 February, 2013

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [14 points] Find explicit general, real-valued solutions for each of the following. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $\frac{dy}{dx} = -\frac{\cos x}{\sin x}\, y + \frac{1}{\sin x}$.

Solution: This is linear and not separable, so we must use an integrating factor. The equation may be rewritten as

$$\frac{dy}{dx} + \frac{\cos x}{\sin x}\, y = \frac{1}{\sin x},$$

so the integrating factor is

$$\mu(x) = e^{\int \frac{\cos x}{\sin x}\, dx} = e^{\ln|\sin x|} = \sin x.$$

Multiplying both sides by μ, we have $(\mu \cdot y)' = 1$, so that $(\sin x)y = x + C$, and therefore

$$y = \frac{x + C}{\sin x}.$$

b. [7 points] $\frac{dy}{dx} - (x-1)y^2 = x - 1$.

Solution: This is nonlinear, but separable. Rewriting the equation, we have $y' = (x-1)(y^2+1)$, so that $\frac{y'}{y^2+1} = x - 1$. Integrating both sides, we have

$$\begin{aligned} \int \frac{y'}{y^2+1}\, dx &= \int x - 1\, dx, \quad \text{or} \\ \arctan(y) &= \frac{1}{2}x^2 - x + C, \quad \text{so} \\ y &= \tan(\frac{1}{2}x^2 - x + C). \end{aligned}$$

2. [14 points] Solve each of the following to find explicit real-valued solutions for y. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] $y'' + 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 4$.

Solution: This is a linear, constant-coefficient problem, so we guess $y = e^{rx}$. Then r satisfies the characteristic equation $r^2 + 6r + 13 = 0$, or $(r+3)^2 + 4 = 0$, so $r = -3 \pm 2i$. Thus the general solution is $y = c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)$. For $y(0) = 0$ we know that $c_1 = 0$, and then $y'(0) = 2c_2 = 4$, so that $c_2 = 2$. Our final solution is $y = 2e^{-3x}\sin(2x)$.

b. [7 points] $y''' + 4y'' = 0$, $y(0) = 0$, $y'(0) = \pi$, $y''(0) = e$.

Solution: This is again a linear constant-coefficient problem, and we guess $y = e^{rx}$. Then we have $r^3 + 4r^2 = r^2(r+4) = 0$, so $r = 0$ (twice) or $r = -4$. The general solution is therefore $y = c_1 + c_2 x + c_3 e^{-4x}$. Plugging in the initial conditions, we have

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 - 4c_3 &= \pi \\ 16c_3 &= e, \end{aligned}$$

so that $c_3 = e/16$, $c_2 = \pi + e/4$ and $c_1 = -e/16$. Our solution is therefore

$$y = -\frac{e}{16} + (\pi + \frac{e}{4})x + \frac{e}{16}e^{-4x}.$$

3. [16 points] Consider the differential equation $ty'' + 2y' = 0$ on the domain $0 < t < \infty$.

a. [6 points] Show that $y_1 = 1$, $y_2 = \frac{1}{t}$ and $y_3 = \frac{t-3}{t}$ are all solutions to this differential equation.

Solution: We can do this by substituting each y_j into the equation. We have $y_1' = y_1'' = 0$, so it is a solution. For y_2 we have $y_2' = -t^{-2}$ and $y_2'' = 2t^{-3}$ so that on plugging in we get

$$2t^{-2} - 2t^{-2} = 0.$$

For y_3 we can note that $y_3 = y_1 - 3y_2$, so by the principle of superposition we know that it is a solution. Alternately, with $y_3 = 1 - \frac{3}{t}$ we have $y_3' = 3t^{-2}$ and $y_3'' = -6t^{-3}$; thus, plugging in, we have

$$-6t^{-2} + 6t^{-2} = 0.$$

Thus all three of these are solutions.

b. [5 points] Are y_1, y_2 and y_3 linearly independent? Explain.

Solution: Above we noted that $y_3 = y_1 - 3y_2$. Thus these are not linearly independent: we can take the linear combination

$$-y_1 + 3y_2 + y_3 = -1 + 3t^{-1} + (1 - 3t^{-1}) = 0.$$

Alternately, we can calculate the Wronskian:

$$W(y_1, y_2, y_3) = \left| \begin{pmatrix} 1 & t^{-1} & 1 - 3t^{-1} \\ 0 & -t^{-2} & 3t^{-2} \\ 0 & 2t^{-3} & -6t^{-6} \end{pmatrix} \right| = 6t^{-6} - 6t^{-6} = 0,$$

so we know that the three are linearly dependent.

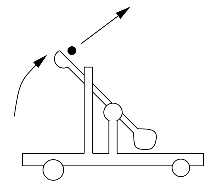
c. [5 points] Write the general solution to this differential equation.

Solution: To write the general solution we need two linearly independent solutions. We can take any pair of y_1, y_2 and y_3 to this end: for example, with y_1 and y_2, we have $W(y_1, y_2) = \left| \begin{pmatrix} 1 & t^{-1} \\ 0 & -t^{-2} \end{pmatrix} \right| = -t^{-2} \neq 0$ (for $t > 0$, the domain on which the equation is defined), so these are linearly independent. Thus a general solution is

$$y = c_1 + c_2 t^{-1}.$$

(Similarly, $W(y_1, y_3) = \left| \begin{pmatrix} 1 & t^{-2} \\ 0 & 2t^{-3} \end{pmatrix} \right| = 2t^{-3} \neq 0$, and $W(y_2, y_3) = \left| \begin{pmatrix} t^{-1} & 1 - 3t^{-1} \\ -t^{-2} & 3t^{-2} \end{pmatrix} \right| = t^{-2} \neq 0$.)

4. [10 points] Suppose we launch a 16 lb bowling ball from a catapult, as suggested in the figure to the right. In this problem we consider the vertical velocity v of the bowling ball. We shall assume that the initial vertical velocity is 45 ft/s, and that the bowling ball is released from a height of 50 ft. Gravity provides a downward acceleration of 32 ft/s^2, and the force of air resistance is proportional to the square of the velocity with constant of proportionality $k = 0.0005$. With these assumptions, the bowling ball reaches its apogee (highest point) of $h = 80.7$ ft at $t = 1.38$ seconds.



a. [6 points] Write an initial value problem for the vertical velocity of the bowling ball on its ascent. Note that you do not need to solve this problem.

Solution: In this case we have $v(0) = 45$. We can set up a differential equation by taking $ma = mv' = \sum(\text{forces})$. The forces are gravity ($= -mg$, so that it points downward) and air resistance ($= -0.0005v^2$, similarly pointing downward), so that

$$mv' = -32m - 0.0005v^2.$$

The mass is $m = 16\text{ lb}/32 = 1/2$ slugs, so that this becomes

$$v' = -32 - 0.001v^2.$$

b. [4 points] Write an initial value problem for the vertical velocity of the bowling ball on its descent. Note that you do not need to solve this problem.

Solution: In this case, the velocity will be negative and so the force of air resistance must point upwards. Our differential equation is

$$v' = -32 + 0.001v^2,$$

and we apply the initial condition $v(1.38) = 0$.

5. [16 points] Consider a differential equation $y' = f(x, y)$ with initial condition $y(0) = 1$. Using two different numerical methods, we obtain the following approximations to the solution of this initial value problem. Note that the error in the approximations is included in the tables.

Method 1:

x	0	0.5	1.0	1.5	2.0
y	1	1	1.1980	1.4238	1.5949
error	0	0.1071	0.1408	0.0794	-0.0358

Method 2:

x	0	0.5	1.0	1.5	2.0
y	1	1.1137	1.3365	1.4558	1.4854
error	0	-0.0066	0.0023	0.0475	0.0736

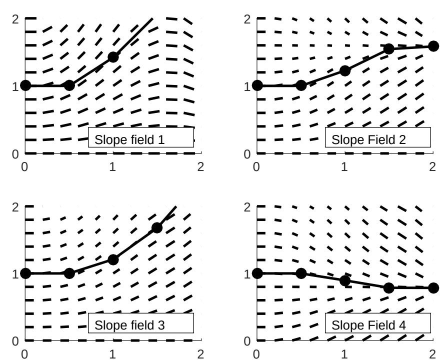
a. [3 points] What is the value of h used in the numerical approximations?

Solution: We note that the increments in x are $\Delta x = 0.5$, so $h = 0.5$.

b. [7 points] One of the methods shown is Euler's method, and the other is improved Euler. Which is which? Why?

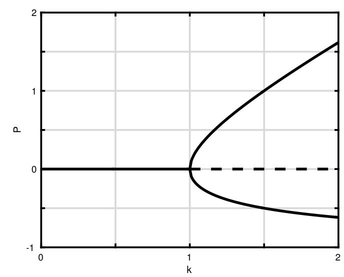
Solution: The first method is Euler's method, and the second improved Euler. We can make a weak argument for this from the given errors: in general we expect the improved Euler method to be more accurate than Euler's method, and for most of the data points this is the case if the first method is Euler's method and the second is the improved Euler method. A strong argument comes if we note that for the first method we have $y(0.5) \approx 1 = y(0)$. For the improved Euler method to give this result, we must have $f(0, 1) = f(0.5, 1) = 0$, in which in which case Euler's method would give $y(0.5) = y(1.0) \approx 1$, which is not the case here. Thus method 1 must be Euler's method, and method 2 the improved Euler method.

c. [6 points] Given the data above, which of the slope fields to the right could be the slope field for this differential equation? Explain.



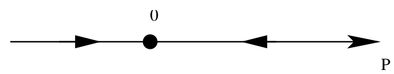
Solution: The correct slope field is slope field 2. We can see this by estimating the Euler's method solution using each slope field, as shown in the figures. For slope fields 1 and 3, the slope fields indicate that Euler's method will overshoot the limiting value shown in the table above; for slope field 4, the slopes are negative, which is not shown in the table. Slope field 2, however, matches the data well.

6. [16 points] Consider a animal population modeled by a differential equation $P' = f(P)$, where the function $f(P)$ involves a parameter k. At $k = 1$ there is a bifurcation point, as shown in the bifurcation diagram to the right. In this figure, solid curves indicate stable solutions while dashed curves indicate unstable ones. Even though $P < 0$ is not physically realizable, include negative values of P in your analysis in parts (a) and (b) below.

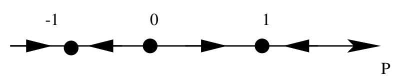


a. [6 points] Sketch phase diagrams for the differential equation $P' = f(P)$ for $k = 0.5$, $k = 1$ and $k = 1.5$.

Solution: We can draw the phase diagrams by using vertical slices of the bifurcation diagram with different values of k. For $k = 0.5$, there is one equilibrium point, $P = 0$, and it is stable, so solutions below and above this must increase and decrease, respectively:

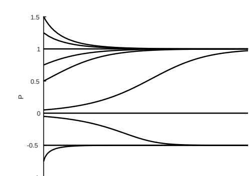


When $k = 1$ we are at the bifurcation point, but still have only the $P = 0$ solution, so the phase diagram is the same. When $k = 1.5$ there are three equilibrium solutions, at $P = -0.5$, $P = 0$ and $P = 1$. The first and last of these are stable, so we know that solutions move toward them. We therefore have the phase diagram shown below.



b. [6 points] Sketch qualitatively reasonable solution curves this equation for the case $k = 1.5$.

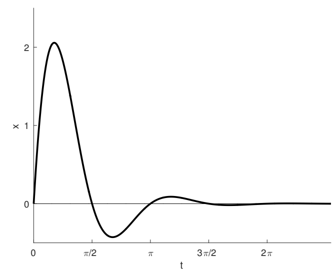
Solution: We know that there are equilibrium solutions at $P = -0.5$, $P = 0$ and $P = 1$. Near any of these the slope of the solution curve will be close to zero; further away, larger. This gives the figure shown to the right. Note that we are unable to say anything about the timescale on which the solutions evolve.



c. [4 points] Thinking of P as an animal population, what is the implication of the bifurcation point? Give a possible explanation for what k could measure.

Solution: The bifurcation point marks the point at which we expect the animals to be able to maintain a non-zero population. For $k \leq 1$, none of the population will survive; for $k > 1$ we expect to see a non-zero stable population. The parameter k could, therefore, model some aspect of the environment related to the carrying capacity. (Other explanations are possible.)

7. [14 points] Consider a spring that, when suspended vertically, is stretched 2 meters by a mass of 2 kg. (For this problem take g, the acceleration due to gravity, to be $g = 10$ m/s^2.) When the mass is attached to the spring and it is positioned horizontally so that the mass can slide back and forth, the motion of the mass is given by



$$2x'' + cx' + kx = 0.$$

a. [3 points] Use the information about the vertical stretch of the spring to find the value of the constant k.

Solution: We know that the force of gravity is equal to the spring force: $mg = F_s = k(\text{stretch})$. Thus $2 \cdot 10 = k \cdot 2$ (because we are taking $g = 10$), so $k = 10$.

b. [7 points] If the mass is set in motion from equilibrium with an initial velocity $v(0) = 8$ m/s, the resulting motion is shown in the figure to the right, above. If we want the motion of the mass to not have any oscillatory characteristics, should we increase or decrease the value of c? If we do this, for what value of c will there first be no oscillation?

Solution: We should increase c (the damping). The first value of c for which there will be no oscillation will be when the roots of the characteristic equation for r go from being complex to being real. The characteristic equation is

$$r^2 + \frac{c}{2}r + 5 = (r + \frac{c}{4})^2 + (5 - \frac{c^2}{16}) = 0.$$

Thus the transition occurs when we have a repeated real root, at $\frac{c^2}{16} = 5$, or $c^2 = 80$. The transition therefore occurs when $c = \sqrt{80} = 4\sqrt{5}$.
Alternately, with the quadratic formula, we have $r = -\frac{c}{4} \pm \frac{1}{2}\sqrt{(c/2)^2 - 20}$, and we go from complex to real roots when $(c/2)^2 = 20$, so $c = 2\sqrt{20} = 4\sqrt{5}$.

c. [4 points] What value of c gives the motion shown in the figure above?

Solution: From the graph we know that the roots of this equation must be complex, so that solutions will look like $x = c_1 e^{-at}\cos(bt) + c_2 e^{-at}\sin(bt)$. With the initial condition $x(0) = 0$ we know that $c_1 = 0$, so that $x = c_2 e^{-at}\sin(bt)$. The graph indicates that this must cross the t-axis at $t = \pi/2$, which means that $b = 2$. Therefore the characteristic equation must be able to be written as $(r + a)^2 + b^2 = (r + a)^2 + 4$. Thus we have

$$r^2 + \frac{c}{2}r + 5 = (r + \frac{c}{4})^2 + (5 - \frac{c^2}{16}) = (r + a)^2 + 4.$$

Comparing the residual terms, $5 - c^2/16 = 4$, so $c^2/16 = 1$ and therefore $c = 4$.
Alternately, using the expression for r that we found above, we have $\frac{1}{2}\sqrt{20 - \frac{c^2}{4}} = 2$, so that $20 - \frac{c^2}{4} = 16$, and $c = 4$, as before.