This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty.** In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [15 points] For this problem note that the general solution to \( y'' + 5y' + 4y = 0 \) is \( y = c_1 e^{-t} + c_2 e^{-4t} \). (Note that minimal partial credit will be given on this problem.)

   a. [7 points] Find a real-valued general solution to

   \[
   y'' + 5y' + 4y = 3e^{-4t}.
   \]

   **Solution:** We know the general solution is \( y = y_c + y_p \). We use the Method of Undetermined Coefficients to find \( y_p \), guessing \( y_p = Ate^{-4t} \), after multiplying our first guess \( (y_p = Ae^{-4t}) \) by \( t \) because the forcing term is present in our homogeneous solution. Then \( y_{p2} = Ae^{-4t} - 4Ate^{-4t} \) and \( y''_{p2} = -8Ae^{-4t} + 4Ate^{-4t} \), so that on plugging in we get

   \[
   (-8A + 5A)e^{-4t} = 3e^{-4t},
   \]

   so that \( -3A = 3 \), and \( A = -1 \).

   Thus the general solution is

   \[
   y = c_1 e^{-t} + c_2 e^{-4t} - te^{-4t}.
   \]

   If we use Variation of Parameters, we have \( u_1'e^{-t} + u_2' e^{-4t} = 0 \) and \( -u_1'e^{-t} - 4u_2'e^{-4t} = 3e^{-4t} \). Solving, we find \( u_2' = -1 \) and \( u_1' = e^{-3t} \), so that \( u_1 = -\frac{1}{3}e^{-3t} \) and \( u_2 = -t \), and \( y_p = -\frac{1}{3}e^{-4t} - te^{-4t} \).

   b. [8 points] Find the solution to the

   \[
   y'' + 5y' + 4y = 16t, \quad y(0) = 2, \quad y'(0) = -2.
   \]

   **Solution:** We know the general solution is \( y = y_c + y_p \). We use the Method of Undetermined Coefficients to find \( y_p \), guessing \( y_p = A + Bt \). Plugging in,

   \[
   5B + 4A + 4Bt = 16t,
   \]

   so that \( B = 4 \) and \( A = -5 \). Thus the general solution is \( y = c_1 e^{-t} + c_2 e^{-4t} - 5 + 4t \).

   Applying the initial conditions, we have

   \[
   y(0) = c_1 + c_2 - 5 = 2, \quad \text{and} \quad y'(0) = -c_1 - 4c_2 + 4 = -2.
   \]

   Thus \( c_1 + c_2 = 7 \) and \( -c_1 - 4c_2 = -6 \). Adding the two, we have \(-3c_2 = 1\), so \( c_2 = -1/3 \). Then the first gives \( c_1 = 22/3 \), and our solution is

   \[
   y = \frac{22}{3} e^{-t} - \frac{1}{3} e^{-4t} - 5 + 4t.
   \]

   We can, of course find \( y_p \) with Variation of Parameters. Then \( u_1'e^{-t} + u_2' e^{-4t} = 0 \) and

   \[
   -u_1'e^{-t} - 4u_2'e^{-4t} = 16t.
   \]

   Solving, we find \( u_2' = -\frac{16}{3} te^{4t} \), so that \( u_2 = (\frac{4}{3} - \frac{2}{3} t) e^{4t} \) and \( u_1' = \frac{16}{3} te^{4t} \), so that \( u_1 = \frac{16}{3} (-1 + t)e^t \). Then \( y_p = -5 + 4t \), as before.
2. [12 points] The eigenvalues of the matrix \( A = \begin{pmatrix} 1 & 5 \\ -5 & 7 \end{pmatrix} \) are \( \lambda = 4 \pm 4i \). Use the eigenvalue method to find a real-valued general solution to the system \( \mathbf{x}' = A\mathbf{x} \). (Note that minimal partial credit will be given on this problem.)

Solution: We know that a complex-valued solution to the system is \( \mathbf{x} = \mathbf{v} e^{(4+4i)t} \), where \( \mathbf{v} \) is the eigenvector corresponding to \( \lambda = 4 + 4i \), and that the real and imaginary parts of this solution will themselves be linearly independent solutions to the system. The eigenvector \( \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} ^T \) is given by

\[
\begin{pmatrix} 1 - \lambda & 5 \\ -5 & 7 - \lambda \end{pmatrix} \mathbf{v} = \begin{pmatrix} -3 - 4i & 5 \\ -5 & 3 - 4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

We know that the two rows of this algebraic system are equivalent, because we are at an eigenvalue. The first gives \((-3 - 4i)v_1 + 5v_2 = 0\), so we may take \( v_1 = 5 \) and \( v_2 = 3 + 4i \). Then a complex-valued solution to the system is

\[
\mathbf{v} e^{(4+4i)t} = \begin{pmatrix} 5 \\ 3 + 4i \end{pmatrix} e^{4t} (\cos(4t) + i\sin(4t))
\]

\[
= \begin{pmatrix} 5 \cos(4t) + 5i\sin(4t) \\ (3\cos(4t) - 4\sin(4t)) + i(4\cos(4t) + 3\sin(4t)) \end{pmatrix} e^{4t}.
\]

Separating the real and imaginary parts of this, we have the general solution

\[
\mathbf{x} = c_1 \begin{pmatrix} 5\cos(4t) \\ 3\cos(4t) - 4\sin(4t) \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 5\sin(4t) \\ 4\cos(4t) + 3\sin(4t) \end{pmatrix} e^{4t}.
\]

Similarly, if we use the second equation to find \( \mathbf{v} \), we have \( \mathbf{v} = \begin{pmatrix} 3 - 4i \\ 5 \end{pmatrix} \), and

\[
\mathbf{x} = c_1 \begin{pmatrix} 3\cos(4t) + 4\sin(4t) \\ 5\cos(4t) \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -4\cos(4t) + 3\sin(4t) \\ 5\sin(4t) \end{pmatrix} e^{4t}.
\]

Using \( \lambda = 4 - 4i \), we get \( \mathbf{v} = \begin{pmatrix} 5 \\ 3 - 4i \end{pmatrix} \) and, reversing the sign of \( c_2 \), the same general solution as before.
3. [16 points] Consider the system
\[ \begin{align*}
    x'_1 &= x_1 + 2x_2 \\
    x'_2 &= 3x_1 
\end{align*} \]

a. [8 points] Find a real-valued general solution to this system.

Solution: Letting \( x = (x_1 \ x_2)^T \), this is equivalent to \( x' = Ax \) with the coefficient matrix \( A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \). Eigenvalues of \( A \) are given by \( \text{det} \left( \begin{pmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{pmatrix} \right) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0 \), so that eigenvalues are \( \lambda = -2 \) and \( \lambda = 3 \). Then eigenvectors \( v \) satisfy the equation
\[ \begin{pmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{pmatrix} v = 0, \]
so that if \( \lambda = -2 \) we have \( v = (-2 \ 3)^T \), and if \( \lambda = 3 \), \( v = (1 \ 1)^T \). Thus the general solution is
\[ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} -2c_1 e^{-2t} + c_2 e^{3t} \\ 3c_1 e^{-2t} + c_2 e^{3t} \end{pmatrix}. \]
This can also be solved by elimination: \( x'_2 = 3x_1 \), so \( x''_2 = 3x'_1 \). Then the first equation becomes \( \frac{1}{3}x''_2 = \frac{1}{3}x'_2 + 2x_2, \) or \( x''_2 - x'_2 - 6x_2 = 0 \). With \( x_2 = e^{rt} \) we have \( r^2 - r - 6 = 0 \), so \( r = -2 \) and \( 3 \). Then \( x_2 = k_1 e^{-2t} + k_2 e^{3t} \). With \( x_1 = \frac{1}{3} x'_2, \) \( x_1 = -\frac{2}{3} k_1 e^{-2t} + k_2 e^{3t} \), which is the same as we found above with \( k_1 = 3c_1 \) and \( k_2 = c_2 \).

b. [4 points] Find the particular solution if \( x_1(0) = 0.10, \ x_2(0) = 0.35 \).

Solution: We have \( x_1(0) = -2c_1 + c_2 = 0.1 \) and \( x_2(0) = 3c_1 + c_2 = 0.35 \). Subtracting the first from the second we get \( c_1 = 0.05 \). Then either equation gives \( c_2 = 0.20 \), and our particular solution is
\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.10 e^{-2t} + 0.20 e^{3t} \\ 0.15 e^{-2t} + 0.20 e^{3t} \end{pmatrix}. \]

\[ \]

c. [4 points] Briefly explain why the direction field and solution trajectory shown to the right could not match this system and your solution from (b).

Solution: In the long run the negative exponentials in the solution we found in (b) will decay to zero and therefore we expect the solution to look like \( x = (0.2 e^{3t} \ 0.2 e^{3t})^T \). Thus the trajectory should end up on the line \( y = x \) (\( x_2 = x_1 \)). This is not shown in the figure to the right. Also note that when \( x_1 = 0 \) the system predicts that trajectories will be horizontal, and when \( x_2 = 0 \) trajectories will have slope 3, neither of which appear to be the case here.
4. [12 points] Three linear constant-coefficient homogeneous systems are described below. Included in the description is one of the following three characteristics; for each, specify the missing two characteristics by circling the correct answers.

1. whether it is a node, saddle point or spiral point, and
2. whether the equilibrium point \((0, 0)\) is asymptotically stable, stable or unstable;
3. the sign of the constant \(a\) in the system.

No explanation is necessary for your answers.

a. [4 points] The system \(\mathbf{x}' = \mathbf{A}\mathbf{x}\) having direction field and representative trajectories in the phase plane shown in the figure to the right, if \(\mathbf{A} = \begin{pmatrix} 1 & -2 \\ a & 1 \end{pmatrix}\) and \(a > 0\).

This is a node and is asymptotically stable. The equilibrium point is stable and \(a > 0\).

Solution: From the figure we see that this is a spiral point. We know the eigenvalues of \(\mathbf{A}\) are given by \((\lambda - 1)^2 + 2a = 0\), so \(\lambda = 1 \pm \sqrt{2a}\), and trajectories increase away from the equilibrium point, indicating that it is unstable.

b. [4 points] The system \(\mathbf{x}' = \mathbf{A}\mathbf{x}\) if the equilibrium point is a saddle point and the eigenvalues are \(\lambda = 3\) and \(\lambda = a\).

The equilibrium point is asymptotically stable. The eigenvalues are given by \((\lambda - 1)^2 + 2a = 0\), so \(\lambda = 1 \pm \sqrt{2a}\), and trajectories increase away from the equilibrium point, indicating that it is unstable.

Solution: Saddle points are unstable, and have real eigenvalues with opposite signs, so \(a < 0\).

c. [4 points] The system \(x_1' = -2x_1 + ax_2, x_2' = x_1 - 2x_2\) whose solution with the initial conditions \(x_1(0) = 0, x_2(0) = 1\) is shown in the figure to the right, if the equilibrium point is asymptotically stable.

The equilibrium point is a node. The eigenvalues are given by \((\lambda - 1)^2 + 2a = 0\), so \(\lambda = 1 \pm \sqrt{2a}\), and trajectories increase away from the equilibrium point, indicating that it is unstable.

Solution: The solution shown has no oscillatory characteristic, so this must be a node with both eigenvalues real and negative. Then the coefficient matrix \(\mathbf{A} = \begin{pmatrix} -2 & a \\ 1 & -2 \end{pmatrix}\), so that eigenvalues are given by \(\text{det}(\mathbf{A} - \lambda I) = (\lambda + 2)^2 - a = 0\). For real eigenvalues, \(\lambda = -2 \pm \sqrt{a}\) must have \(a > 0\) (and \(a < 4\), for that matter).
5. [16 points] Consider a clown on a spring in a boat, as suggested by the figure to the right. At time \( t = 0 \) we place a large box in the boat. Then, with some not entirely unreasonable assumptions, the displacement \( x_1 \) of the boat and \( x_2 \) of the clown are given by

\[
\begin{align*}
  x''_1 &= -425x_1 + 75x_2 - 35 \\
  x''_2 &= 150x_1 - 150x_2.
\end{align*}
\]

(Here, \( x_1 \) and \( x_2 \) are measured in meters and \( t \) in seconds.) Letting \( A = \begin{pmatrix} -425 & 75 \\ 150 & -150 \end{pmatrix} \), the eigenvalues and eigenvectors of \( A \) are \( \lambda = -600 \) and \( \lambda = -125 \) with \( \mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \) and \( \mathbf{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \).

a. [6 points] What are the natural frequencies at which the boat and clown will oscillate? Explain.

\[
\text{Solution:} \quad \text{Letting } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } A = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}, \text{ we guess } \mathbf{x} = \mathbf{v} e^{rt}, \text{ so that } r^2 \mathbf{v} = A \mathbf{v}, \text{ and therefore } r^2 = \lambda, \text{ the eigenvalues of } A. \text{ Thus } r = \pm i\sqrt{600} \text{ or } r = \pm i\sqrt{125}, \text{ and solutions will look like cosines and sines of } \sqrt{600} t \text{ and } \sqrt{125} t. \text{ Thus the frequencies are } \omega_1 = \sqrt{600}(\approx 24) \text{ and } \omega_2 = \sqrt{125}(\approx 11). (Note that if we were recently indoctrinated by some other field we might also talk about the ordinary frequency, } f = \omega/2\pi.\]

b. [6 points] Find the general solution to the homogeneous system associated with this system.

\[
\text{Solution:} \quad \text{Given the eigenvalues and eigenvectors provided, we have}
\]

\[
\mathbf{x}(t) = (c_1 \cos(\sqrt{600} t) + c_2 \sin(\sqrt{600} t)) \begin{pmatrix} -3 \\ 1 \end{pmatrix} + (c_3 \cos(\sqrt{125} t) + c_4 \sin(\sqrt{125} t)) \begin{pmatrix} 1 \\ 6 \end{pmatrix}.
\]

c. [4 points] A solution to this system is shown to the right. What initial conditions were applied to \( x_1 \) and \( x_2 \) to obtain this solution?

\[
\text{Solution:} \quad \text{The initial conditions were } x_1(0) = 0.1, \text{ and } x'_1(0) = x_2(0) = x'_2(0) = 0. \text{ We see that } x_1 \text{ starts at 0.1, } x_2 \text{ at 0, and that each solution curve starts at 0 with zero slope.}\]
6. [15 points] Consider a mass-spring system modeled by

\[ y'' + cy' + ky = f(t), \]

where \( c \) is the damping coefficient associated with the system and \( k \) the spring constant. For each of the following give a short explanation of your answers.

a. [5 points] If \( f(t) = 3 \sin(2t) \) and the system is at resonance, are \( c \) and \( k \) positive, negative or zero? Give specific values for \( c \) and \( k \) if possible.

**Solution:** If the system experiences resonance, we know that \( c = 0 \). To have resonance with a forcing that has a frequency 2, we know that the natural frequency of the system, which is \( \omega = \sqrt{k} \) must be 2, so that \( k = 4 \).

b. [5 points] If \( f(t) = 3 \sin(2t) \) and the system is at resonance, sketch a qualitatively accurate graph of \( y_p \), the particular solution to the problem.

**Solution:** Because the system is at resonance, we know that \( y = At \cos(2t) \) (we may omit the \( t \sin(2t) \) term because there are only even derivatives in the problem). Thus the solution will be something like the following graph.

![Graph of y_p](image)

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c. [5 points] If \( c > 0 \), \( k > 0 \), and \( f(t) = 3 \sin(2t) \), what can you say (without solving the differential equation) about the long-term behavior of \( y \)?

**Solution:** If \( c > 0 \) the homogeneous solutions will be decaying. Thus the long-term response will be \( y_p \), the particular solution, which will be \( y_p = A \cos(2t) + B \sin(2t) = \sqrt{A^2 + B^2} \cos(2t - \alpha) \) for some \( A \) and \( B \). That is, the long-term response will be purely sinusoidal with period \( \pi \).
7. [14 points] In this problem we consider the system
\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}' = \begin{pmatrix} 0 & t/2 \\ -4t^{-3} & t^{-1}
\end{pmatrix} \begin{pmatrix} x_1 \\
x_2
\end{pmatrix},
\]
with the initial condition \( \begin{pmatrix} x_1(1) \\
x_2(1)
\end{pmatrix} = \begin{pmatrix} a \\
b
\end{pmatrix} (a, b \neq 0)\).

a. [4 points] Find the Euler’s method approximation for the solution to the system after one step with a step size \( h \) (your answer will involve \( a, b \) and \( h \)). What is the meaning of your result?

\textit{Solution:} After one step we have
\[
x \approx \begin{pmatrix} a \\
b
\end{pmatrix} + h \begin{pmatrix} 0 & \frac{1}{2} \\ -4 & 1
\end{pmatrix} \begin{pmatrix} a \\
b
\end{pmatrix} = \begin{pmatrix} a + \frac{1}{2}hb \\
b + h(-4a + b)
\end{pmatrix}.
\]
This is the approximation for \( x \) at \( t = 1 + h \).

b. [4 points] Rewrite your approximation in the form \( P \begin{pmatrix} a \\
b
\end{pmatrix} \). What is \( P \)?

\textit{Solution:} The matrix \( P \) is the coefficient matrix from the approximation,
\[
P = \begin{pmatrix} 1 & \frac{1}{2}h \\
-4h & (1 + h)
\end{pmatrix}.
\]

c. [2 points] Find \( \det(P) \).

\textit{Solution:} \( \det(P) = 1 + h + 2h^2 \).

d. [4 points] Is it possible that the Euler step could end at \( x_1 = 0, x_2 = 0 \)? Explain.

\textit{Solution:} Note that \( \det(P) = 2h^2 + h + 1 \neq 0 \). Therefore \( P \) is nonsingular (has an inverse), and the only way that this approximation, \( P \begin{pmatrix} a \\
b
\end{pmatrix} \), could equal the zero vector is if \( a = b = 0 \). Because we know that \( a, b \neq 0 \), this is therefore not possible.

We could also work this out directly: we have the approximations \( x_1(1 + h) = a + \frac{1}{2}hb \) and \( x_2(1 + h) = -4ah + (1 + h)b \). Setting both to zero requires that \( a + \frac{1}{2}hb = 0 \), so that \( a = -\frac{1}{2}hb \), and then that \( -4ah + (1 + h)b = 2h^2b + (1 + h)b = (2h^2 + h + 1)b = 0 \). We know \( b \neq 0 \), and \( 2h^2 + h + 1 \neq 0 \), so this is not possible. Note that this condition is, not surprisingly, the same as the determinant.