Math 216 — Second Midterm

28 March, 2013

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [15 points] For this problem note that the general solution to $y'' + 5y' + 4y = 0$ is $y = c_1e^{-t} + c_2e^{-4t}$. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Find a real-valued general solution to

$$y'' + 5y' + 4y = 3e^{-4t}.$$

Solution: We know the general solution is $y = y_c + y_p$. We use the Method of Undetermined Coefficients to find y_p, guessing $y_p = Ate^{-4t}$, after multiplying our first guess ($y_p = Ae^{-4t}$) by t because the forcing term is present in our homogeneous solution. Then $y'_{p2} = Ae^{-4t} - 4Ate^{-4t}$ and $y''_{p2} = -8Ae^{-4t} + 4Ate^{-4t}$, so that on plugging in we get

$$(-8A + 5A)e^{-4t} = 3e^{-4t},$$

so that $-3A = 3$, and $A = -1$.
Thus the general solution is

$$y = c_1e^{-t} + c_2e^{-4t} - te^{-4t}.$$

If we use Variation of Parameters, we have $u'_1e^{-t} + u'_2e^{-4t} = 0$ and $-u'_1e^{-t} - 4u'_2e^{-4t} = 3e^{-4t}$. Solving, we find $u'_2 = -1$ and $u'_1 = e^{-3t}$, so that $u_1 = -\frac{1}{3}e^{-3t}$ and $u_2 = -t$, and $y_p = -\frac{1}{3}e^{-4t} - te^{-4t}$.

b. [8 points] Find the solution to the

$$y'' + 5y' + 4y = 16t, \qquad y(0) = 2, \quad y'(0) = -2.$$

Solution: We know the general solution is $y = y_c + y_p$. We use the Method of Undetermined Coefficients to find y_p, guessing $y_p = A + Bt$. Plugging in,

$$5B + 4A + 4Bt = 16t,$$

so that $B = 4$ and $A = -5$. Thus the general solution is $y = c_1e^{-t} + c_2e^{-4t} - 5 + 4t$.
Applying the initial conditions, we have

$$\begin{aligned} y(0) &= c_1 + c_2 - 5 = 2, \quad \text{and} \\ y'(0) &= -c_1 - 4c_2 + 4 = -2. \end{aligned}$$

Thus $c_1 + c_2 = 7$ and $-c_1 - 4c_2 = -6$. Adding the two, we have $-3c_2 = 1$, so $c_2 = -1/3$. Then the first gives $c_1 = 22/3$, and our solution is

$$y = \frac{22}{3}\,e^{-t} - \frac{1}{3}\,e^{-4t} - 5 + 4t.$$

We can, of course find y_p with Variation of Parameters. Then $u'_1e^{-t} + u'_2e^{-4t} = 0$ and $-u'_1e^{-t} - 4u'_2e^{-4t} = 16t$. Solving, we find $u'_2 = -\frac{16}{3}te^{4t}$, so that $u_2 = (\frac{1}{3} - \frac{4}{3}t)e^{4t}$ and $u'_1 = \frac{16}{3}te^{t}$, so that $u_1 = \frac{16}{3}(-1 + t)e^{t}$. Then $y_p = -5 + 4t$, as before.

2. [12 points] The eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ -5 & 7 \end{pmatrix}$ are $\lambda = 4 \pm 4i$. Use the eigenvalue method to find a real-valued general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. *(Note that minimal partial credit will be given on this problem.)*

Solution: We know that a complex-valued solution to the system is $\mathbf{x} = \mathbf{v}e^{(4+4i)t}$, where $\mathbf{v}$ is the eigenvector corresponding to $\lambda = 4 + 4i$, and that the real and imaginary parts of this solution will themselves be linearly independent solutions to the system. The eigenvector $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ is given by

$$\begin{pmatrix} 1-\lambda & 5 \\ -5 & 7-\lambda \end{pmatrix} \mathbf{v} = \begin{pmatrix} -3-4i & 5 \\ -5 & 3-4i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We know that the two rows of this algebraic system are equivalent, because we are at an eigenvalue. The first gives $(-3-4i)v_1 + 5v_2 = 0$, so we may take $v_1 = 5$ and $v_2 = 3 + 4i$. Then a complex-valued solution to the system is

$$\begin{aligned} \mathbf{v}e^{(4+4i)t} &= \begin{pmatrix} 5 \\ 3+4i \end{pmatrix} e^{4t}(\cos(4t) + i\sin(4t)) \\ &= \begin{pmatrix} 5\cos(4t) + 5i\sin(4t) \\ (3\cos(4t) - 4\sin(4t)) + i(4\cos(4t) + 3\sin(4t)) \end{pmatrix} e^{4t}. \end{aligned}$$

Separating the real and imaginary parts of this, we have the general solution

$$\mathbf{x} = c_1 \begin{pmatrix} 5\cos(4t) \\ 3\cos(4t) - 4\sin(4t) \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 5\sin(4t) \\ 4\cos(4t) + 3\sin(4t) \end{pmatrix} e^{4t}.$$

Similarly, if we use the second equation to find $\mathbf{v}$, we have $\mathbf{v} = \begin{pmatrix} 3-4i \\ 5 \end{pmatrix}$, and

$$\mathbf{x} = c_1 \begin{pmatrix} 3\cos(4t) + 4\sin(4t) \\ 5\cos(4t) \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -4\cos(4t) + 3\sin(4t) \\ 5\sin(4t) \end{pmatrix} e^{4t}.$$

Using $\lambda = 4 - 4i$, we get $\mathbf{v} = \begin{pmatrix} 5 \\ 3-4i \end{pmatrix}$ and, reversing the sign of c_2, the same general solution as before.

3. [16 points] Consider the system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1$$

a. [8 points] Find a real-valued general solution to this system.

Solution: Letting $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$, this is equivalent to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with the coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$. Eigenvalues of $\mathbf{A}$ are given by $\det \begin{pmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2) = 0$, so that eigenvalues are $\lambda = -2$ and $\lambda = 3$. Then eigenvectors $\mathbf{v}$ satisfy the equation

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{pmatrix} \mathbf{v} = \mathbf{0},$$

so that if $\lambda = -2$ we have $\mathbf{v} = \begin{pmatrix} -2 & 3 \end{pmatrix}^T$, and if $\lambda = 3$, $\mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$. Thus the general solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} -2c_1 e^{-2t} + c_2 e^{3t} \\ 3c_1 e^{-2t} + c_2 e^{3t} \end{pmatrix}.$$

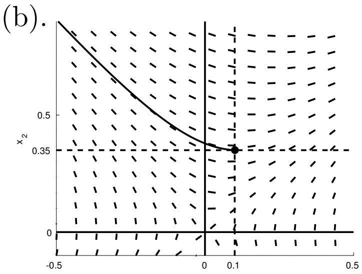
This can also be solved by elimination: $x_2' = 3x_1$, so $x_2'' = 3x_1'$. Then the first equation becomes $\frac{1}{3}x_2'' = \frac{1}{3}x_2' + 2x_2$, or $x_2'' - x_2' - 6x_2 = 0$. With $x_2 = e^{rt}$ we have $r^2 - r - 6 = 0$, so $r = -2$ and 3. Then $x_2 = k_1 e^{-2t} + k_2 e^{3t}$. With $x_1 = \frac{1}{3}x_2'$, $x_1 = -\frac{2}{3}k_1 e^{-2t} + k_2 e^{3t}$, which is the same as we found above with $k_1 = 3c_1$ and $k_2 = c_2$.

b. [4 points] Find the particular solution if $x_1(0) = 0.10$, $x_2(0) = 0.35$.

Solution: We have $x_1(0) = -2c_1 + c_2 = 0.1$ and $x_2(0) = 3c_1 + c_2 = 0.35$. Subtracting the first from the second we get $c_1 = 0.05$. Then either equation gives $c_2 = 0.20$, and our particular solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.10\, e^{-2t} + 0.20\, e^{3t} \\ 0.15\, e^{-2t} + 0.20\, e^{3t} \end{pmatrix}.$$

c. [4 points] Briefly explain why the direction field and solution trajectory shown to the right could not match this system and your solution from (b).



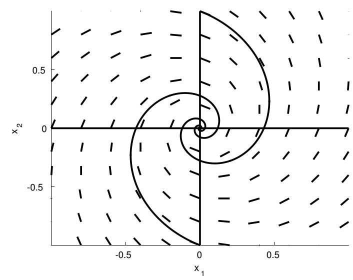
Solution: In the long run the negative exponentials in the solution we found in (b) will decay to zero and therefore we expect the solution to look like $\mathbf{x} = \begin{pmatrix} 0.2e^{3t} & 0.2e^{3t} \end{pmatrix}^T$. Thus the trajectory should end up on the line $y = x$ ($x_2 = x_1$). This is not shown in the figure to the right. Also note that when $x_1 = 0$ the system predicts that trajectories will be horizontal, and when $x_2 = 0$ trajectories will have slope 3, neither of which appear to be the case here.

4. [12 points] Three linear constant-coefficient homogeneous systems are described below. Included in the description is one of the following three charateristics; for each, specify the missing two characteristics by circling the correct answers.

1. whether it is a node, saddle point or spiral point, and
2. whether the equilibrium point $(0, 0)$ is asymptotically stable, stable or unstable;
3. the sign of the constant a in the system.

No explanation is necessary for your answers.

a. [4 points] The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ having direction field and representative trajectories in the phase plane shown in the figure to the right, if $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ a & 1 \end{pmatrix}$ and $a > 0$.



This is a

node   saddle point   [spiral point]

and is

asymptotically stable   stable   [unstable]

Solution: From the figure we see that this is a **spiral point**. We know the eigenvalues of $\mathbf{A}$ are given by $(\lambda - 1)^2 + 2a = 0$, so $\lambda = 1 \pm \sqrt{2a}i$, and trajectories increase away from the equilibrium point, indicating that it is **unstable**.

b. [4 points] The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ if the equilibrium point is a saddle point and the eigenvalues are $\lambda = 3$ and $\lambda = a$.

The equilibrium point is

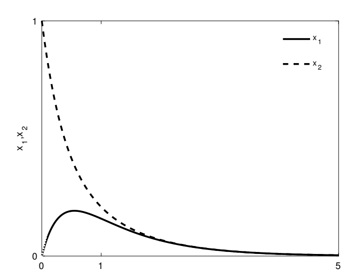
asymptotically stable   stable   [unstable]

and a is

[< 0] > 0

Solution: Saddle points are **unstable**, and have real eigenvalues with opposite signs, so $a < 0$.

c. [4 points] The system $x_1' = -2x_1 + ax_2$, $x_2' = x_1 - 2x_2$ whose solution with the initial conditions $x_1(0) = 0$, $x_2(0) = 1$ is shown in the figure to the right, if the equilibrium point is asymptotically stable.



The equilibrium point is a

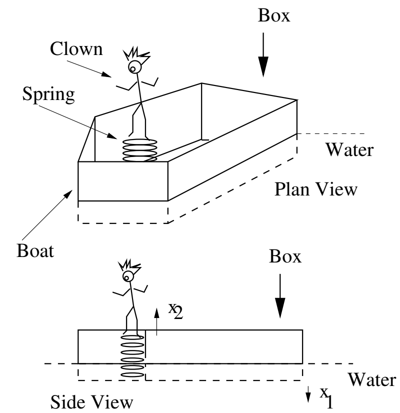
[node]   saddle point   spiral point

and a is

< 0 [> 0]

Solution: The solution shown has no oscillatory characteristic, so this must be a **node** with both eigenvalues real and negative. Then the coefficient matrix $\mathbf{A} = \begin{pmatrix} -2 & a \\ 1 & -2 \end{pmatrix}$, so that eigenvalues are given by $\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda + 2)^2 - a = 0$. For real eigenvalues, $\lambda = -2 \pm \sqrt{a}$ must have $a > 0$ (and $a < 4$, for that matter).

5. [16 points] Consider a clown on a spring in a boat, as suggested by the figure to the right. At time $t = 0$ we place a large box in the boat. Then, with some not entirely unreasonable assumptions, the displacement x_1 of the boat and x_2 of the clown are given by



$$x_1'' = -425x_1 + 75x_2 - 35$$
$$x_2'' = 150x_1 - 150x_2.$$

(Here, x_1 and x_2 are measured in meters and t in seconds.) Letting $\mathbf{A} = \begin{pmatrix} -425 & 75 \\ 150 & -150 \end{pmatrix}$, the eigenvalues and eigenvectors of $\mathbf{A}$ are $\lambda = -600$ and $\lambda = -125$ with $\mathbf{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$.

a. [6 points] What are the natural frequencies at which the boat and clown will oscillate? Explain.

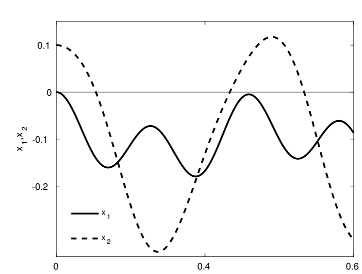
Solution: Letting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$, we guess $\mathbf{x} = \mathbf{v}e^{rt}$, so that $r^2\mathbf{v} = \mathbf{A}\mathbf{v}$, and therefore $r^2 = \lambda$, the eigenvalues of $\mathbf{A}$. Thus $r = \pm i\sqrt{600}$ or $r = \pm i\sqrt{125}$, and solutions will look like cosines and sines of $\sqrt{600}\,t$ and $\sqrt{125}\,t$. Thus the frequencies are $\omega_1 = \sqrt{600}(\approx 24)$ and $\omega_2 = \sqrt{125}(\approx 11)$. (Note that if we were recently indoctrinated by some other field we might also talk about the ordinary frequency, $f = \omega/2\pi$.)

b. [6 points] Find the general solution to the homogeneous system associated with this system.

Solution: Given the eigenvalues and eigenvectors provided, we have

$$\mathbf{x}_c = (c_1 \cos(\sqrt{600}\,t) + c_2 \sin(\sqrt{600}\,t)) \begin{pmatrix} -3 \\ 1 \end{pmatrix} + (c_3 \cos(\sqrt{125}\,t) + c_4 \sin(\sqrt{125}\,t)) \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

c. [4 points] A solution to this system is shown to the right. What initial conditions were applied to x_1 and x_2 to obtain this solution?



Solution: The initial conditions were $x_1(0) = 0.1$, and $x_1'(0) = x_2(0) = x_2'(0) = 0$. We see that x_1 starts at 0.1, x_2 at 0, and that each solution curve starts at 0 with zero slope.

6. [15 points] Consider a mass-spring system modeled by

$$y'' + cy' + ky = f(t),$$

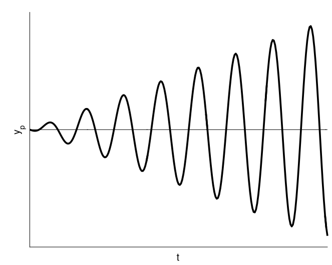
where c is the damping coefficient associated with the system and k the spring constant. For each of the following give a short explanation of your answers.

a. [5 points] If $f(t) = 3\sin(2t)$ and the system is at resonance, are c and k positive, negative or zero? Give specific values for c and k if possible.

Solution: If the system experiences resonance, we know that $c = 0$. To have resonance with a forcing that has a frequency 2, we know that the natural frequency of the system, which is $\omega = \sqrt{k}$ must be 2, so that $k = 4$.

b. [5 points] If $f(t) = 3\sin(2t)$ and the system is at resonance, sketch a qualitatively accurate graph of y_p, the particular solution to the problem.

Solution: Because the system is at resonance, we know that $y = A\,t\,\cos(2t)$ (we may omit the $t\sin(2t)$ term because there are only even derivatives in the problem). Thus the solution will be something like the following graph.



c. [5 points] If $c > 0$, $k > 0$, and $f(t) = 3\sin(2t)$, what can you say (without solving the differential equation) about the long-term behavior of y?

Solution: If $c > 0$ the homogeneous solutions will be decaying. Thus the long-term response will be y_p, the particular solution, which will be $y_p = A\cos(2t) + B\sin(2t) = \sqrt{A^2 + B^2}\cos(2t - \alpha)$ for some A and B. That is, the long-term response will be purely sinusoidal with period π.

7. [14 points] In this problem we consider the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & t/2 \\ -4t^{-3} & t^{-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

with the initial condition $\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ $(a, b \neq 0)$.

a. [4 points] Find the Euler's method approximation for the solution to the system after one step with a step size h (your answer will involve a, b and h). What is the meaning of your result?

Solution: After one step we have

$$\mathbf{x} \approx \begin{pmatrix} a \\ b \end{pmatrix} + h \begin{pmatrix} 0 & \frac{1}{2} \\ -4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + \frac{1}{2}hb \\ b + h(-4a + b) \end{pmatrix}.$$

This is the approximation for $\mathbf{x}$ at $t = 1 + h$.

b. [4 points] Rewrite your approximation in the form $\mathbf{P}\begin{pmatrix} a \\ b \end{pmatrix}$. What is $\mathbf{P}$?

Solution: The matrix $\mathbf{P}$ is the coefficient matrix from the approximation,

$$\mathbf{P} = \begin{pmatrix} 1 & \frac{1}{2}h \\ -4h & (1+h) \end{pmatrix}.$$

c. [2 points] Find $\det(\mathbf{P})$.

Solution: $\det(\mathbf{P}) = 1 + h + 2h^2$.

d. [4 points] Is it possible that the Euler step could end at $x_1 = 0$, $x_2 = 0$? Explain.

Solution: Note that $\det(\mathbf{P}) = 2h^2 + h + 1 \neq 0$. Therefore $\mathbf{P}$ is nonsingular (has an inverse), and the only way that this approximation, $\mathbf{P}\begin{pmatrix} a \\ b \end{pmatrix}$, could equal the zero vector is if $a = b = 0$. Because we know that $a, b \neq 0$, this is therefore not possible.
We could also work this out directly: we have the approximations $x_1(1+h) = a + \frac{1}{2}hb$ and $x_2(1+h) = -4ah + (1+h)b$. Setting both to zero requires that $a + \frac{1}{2}hb = 0$, so that $a = -\frac{1}{2}hb$, and then that $-4ah + (1+h)b = 2h^2b + (1+h)b = (2h^2 + h + 1)b = 0$. We know $b \neq 0$, and $2h^2 + h + 1 \neq 0$, so this is not possible. Note that this condition is, not surprisingly, the same as the determinant.