Math 216 — Final Exam

26 April, 2013

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Solve each of the following, as indicated.
Note that little partial credit will be given in this problem.

a. [6 points] Find an explicit solution to $x^2\, y\, y' = y^2 + 1$, $y(1) = 1$.

Solution: This is first-order and separable, and nonlinear, so our only solution method is to use separation of variables. Separating, we have

$$\frac{y}{y^2+1}\, y' = \frac{1}{x^2},$$

so that, after integrating, $\frac{1}{2}\ln(y^2+1) = -\frac{1}{x} + C$. Solving for y, we have

$$y = \pm\sqrt{-1 + Ce^{-2/x}}.$$

Applying the initial condition $y(1) = 1$, we take the positive branch and let $C = 2e^2$, so that

$$y = \sqrt{-1 + 2e^{2(1-1/x)}}.$$

b. [6 points] Find a general real-valued solution to $y''' - y'' + 4y' - 4y = 3e^t$ if one solution to the associated homogeneous equation is $y = e^t$.

Solution: Our general solution will be $y = y_c + y_p$, where y_c and y_p are, respectively, the solution to the complementary homogeneous and full problems. For y_c, we guess $y = e^{rt}$, getting $r^3 - r^2 + 4r - 4 = r^2(r-1) + 4(r-1) = (r^2+4)(r-1) = 0$. Thus $r = 1$ and $r = \pm 2i$, and our solution is

$$y_c = c_1\cos(2t) + c_2\sin(2t) + c_3e^t.$$

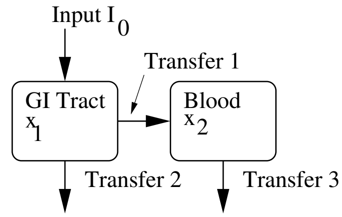
For y_p, we would guess $y_p = Ae^t$, but this is part of the homogeneous solution, so we guess $y_p = Ate^t$ instead. Then $y_p = A(1+t)e^t$, $y_p'' = A(2+t)e^t$ and $y_p''' = A(3+t)e^t$. Plugging in, we have

$$A(3 + t - 2 - t + 4 + 4t - 4t)e^t = 5Ae^t = 3e^t,$$

so that $A = 3/5$, and our general solution is

$$y = c_1\cos(2t) + c_2\sin(2t) + c_3e^t + \frac{3}{5}te^t.$$

2. [12 points] Consider the two compartment system shown in the figure to the right. We suppose that it models a drug taken orally, so that the input I_0 enters the gastrointestinal tract, resulting in an amount x_1 of the drug there. This is transferred to the blood and removed at rates proportional to the amount present, as suggested by the figure. The amount of the drug in the blood is then x_2, and this is reduced at a rate proportional to the amount present as well.



a. [2 points] Explain why this results in the system of equations

$$x_1' = -(k_1 + k_2)x_1 + I_0$$
$$x_2' = k_1 x_1 - k_3 x_2$$

Solution: For both compartments, we have rate of change = rate in − rate out. For the first compartment, the input is I_0 and outputs are k_1x_1 to the second compartment and k_2x_2 from the system, so that $x_1' = I - (k_1 + k_2)x_1$. For the second compartment, the input is k_1x_1 and output is k_3x_2, so $x_2' = k_1x_1 - k_3x_2$.

b. [4 points] Suppose that with some appropriate assumptions $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, and $I_0 = 12$. Take $x_1(0) = x_2(0) = 0$ and solve the system by finding x_1 and using this to find x_2.

Solution: The first equation is $x_1' + 4x_1 = 12$. Using integrating factors, we have $(e^{4t}x_1)' = 12e^{4t}$, so that $x_1 = C_1e^{-4t} + 3$. If $x_1(0) = 0$, this is

$$x_1 = 3(1 - e^{-4t}).$$

Then $x_2' + 2x_2 = 3(1 - e^{-4t})$, so, again using integrating factors, $(e^{2t}x_2)' = 3e^{2t} - 3e^{-2t}$. Integrating and solving for x_2, we get $x_2 = \frac{3}{2}(1 + e^{-4t}) + C_2e^{-2t}$. Applying the initial condition $x_2(0) = 0$, we have

$$x_2 = \frac{3}{2}(1 + e^{-4t}) - 3e^{-2t}.$$

Problem 2, continued. We are considering the system

$$x_1' = -4x_1 + 12, \qquad x_2' = x_1 - 2x_2.$$

c. [3 points] This system can be written as a linear second-order constant-coefficient equation for x_2. Based on your solution in (b), what is the characteristic equation for this second order equation?

Solution: We know that the equation will be $x_2'' + ax_2' + bx_2 = f(t)$, and that the homogeneous solution is therefore $x_{2c} = c_1 e^{r_1 t} + c_2 e^{r_2 t}$. From our work in (a), we know that $r_1 = 4$ and $r_2 = 2$; thus the characteristic equation must be $(r + 4)(r + 2) = r^2 + 6r + 8 = 0$.

d. [3 points] Finally, write the system as a 2×2 system of equations in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}+\mathbf{f}$.

Solution: We have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \mathbf{x}' = \begin{pmatrix} -4 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}.$$

3. [12 points] Consider a grand stage show put on to celebrate the conclusion of your math 216 experience: hundreds of magicians crowd a stage, and every minute,

- 20% of the magicians leave the stage;
- each magician produces—*from thin air*—one rabbit and 4 flowers;
- each rabbit eats 2% of the flowers on the stage, and 10% of the rabbits hop off-stage in search of other food.

With x_1 = the number of magicians on stage, x_2 = the number of rabbits, and x_3 = the number of flowers, this leads to the system of equations

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} -1/5 & 0 & 0 \\ 1 & -1/10 & 0 \\ 4 & 0 & -1/50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Find the general real-valued solution to this problem using the eigenvalue method.

Solution: Let $\mathbf{x} = \mathbf{v}e^{\lambda t}$. Then the λs are the eigenvalues of the coefficient matrix, which we can read from the diagonal of the matrix: $\lambda = -1/5$, $-1/10$, and $-1/50$. For each we find the eigenvectors by letting $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^T$ and solving the equation $(\mathbf{A} - \lambda I)\mathbf{v} = \mathbf{0}$.
If $\lambda = -1/5$, the first row of this equation is trivial, and we have for the remaining two $v_1 + (1/10)v_2 = 0$ and $4v_1 + (9/50)v_3 = 0$, and so may pick $\mathbf{v} = \begin{pmatrix} -9 \\ 90 \\ 200 \end{pmatrix}$.
If $\lambda = -1/10$, we have $-(1/10)v_1 = 0$, $v_1 = 0$ and $4v_1 + (4/50)v_3 = 0$. Thus $v_1 = v_3 = 0$, and we may pick $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Finally, if $\lambda = -1/50$, we have $v_1 = v_2 = 0$ and may pick $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Thus the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 \begin{pmatrix} -9 \\ 90 \\ 200 \end{pmatrix} e^{-t/5} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t/10} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t/50}.$$

4. [12 points] Find each of the indicated Laplace or inverse Laplace transforms. Note that if the problem indicates that you should use the definition of the transform, looking up the solution in the table at the end of the exam will be worth zero points.

a. [4 points] Find $\mathfrak{L}^{-1}\{\frac{2}{(s^2+9)(s+2)}\}$

Solution: Using partial fractions, we have $\frac{2}{(s^2+9)(s+2)} = \frac{As+B}{s^2+9} + \frac{C}{s+2}$, so that $2 = (As+B)(s+2) + C(s^2+9)$. Letting $s = -2$, we have $C = 2/13$. Then if $s = 0$, $2B = 2 - \frac{18}{13} = \frac{8}{13}$, so that $B = 4/13$, and finally, matching the coefficients of s^2, we have $A = -C = -2/13$. Thus we have $-\frac{2}{13}\mathfrak{L}^{-1}\{\frac{s}{s^2+9}\} + \frac{4}{13}\mathfrak{L}^{-1}\{\frac{1}{s^2+9}\} + \frac{2}{13}\mathfrak{L}^{-1}\{\frac{1}{s+2}\} = -\frac{2}{13}\cos(3t) + \frac{4}{39}\sin(3t) + \frac{2}{13}e^{-2t}$.

b. [4 points] Find $\mathfrak{L}^{-1}\{\frac{s}{(s^2+4)^2}\}$

Solution: With $F(s) = \frac{1}{s^2+4}$ (so that $f(t) = \frac{1}{2}\sin(2t)$), note that $-\frac{1}{2}F'(s) = \frac{s}{(s^2+4)^2}$. Thus $\mathfrak{L}^{-1}\{\frac{s}{(s^2+4)^2}\} = -\frac{1}{2}\mathfrak{L}^{-1}\{F'(s)\} = -\frac{1}{2}(-t\cdot\frac{1}{2}\sin(2t)) = \frac{1}{4}\,t\,\sin(2t)$.

Alternately, with $F(s) = \frac{s}{(s^2+4)^2}$, we have $\mathfrak{L}^{-1}\{\int_s^\infty F(\sigma)\,d\sigma\} = \mathfrak{L}^{-1}\{\lim_{b\to\infty} -\frac{1}{2}\frac{1}{\sigma^2+4}\Big|_s^b\} = \mathfrak{L}^{-1}\{\frac{1}{2}\frac{1}{s^2+4}\} = \frac{1}{4}\sin(2t) = \frac{1}{t}\,f(t)$. Thus $f(t) = \mathfrak{L}^{-1}\{F(s)\} = \frac{1}{4}\,t\sin(2t)$.

c. [4 points] Use the definition of the Laplace transform to show that $\mathfrak{L}\{f'(t)\} = sF(s) - f(0)$, where $\mathfrak{L}\{f(t)\} = F(s)$.

Solution: We have, using integration by parts with $u = e^{-st}$ and $v' = f'$, $\mathfrak{L}\{f'(t)\} = \int_0^\infty f'(t)\,e^{-st}\,dt = \lim_{b\to\infty} f(t)\,e^{-st}\Big|_{t=0}^b + \int_0^\infty se^{-st}f(t)\,dt = -f(0) + s\int_0^\infty e^{-st}\,f(t)\,dt = -f(0) + sF(s)$.

5. [14 points] Solve using Laplace transforms:

$$y'' + 2y' + y = e^{-t}(1 - u(t-2)), \qquad y(0) = 3, \quad y'(0) = 4.$$

Solution: Note that $e^{-t}\,u(t-2) = e^{-2}\,e^{-(t-2)}\,u(t-2)$. Then we can forward transform using known rules to get, with $\mathfrak{L}\{y(t)\} = Y(s)$,

$$s^2 Y - 3s - 4 + 2sY - 6 + Y = \frac{1}{s+1} - \frac{e^{-2}\,e^{-2s}}{s+1},$$

so that

$$(s+1)^2 Y = 3s + 10 - \frac{1}{s+1} - \frac{e^{-2}\,e^{-2s}}{s+1}.$$

Dividing by $(s+1)^2$ and rewriting $3s+10$ as $3s+3+7$, we have

$$Y = \frac{3s+10}{(s+1)^2} - \frac{1}{(s+1)^3} - \frac{e^{-2}e^{-2s}}{(s+1)^3} = \frac{3}{s+1} + \frac{7}{(s+1)^2} - \frac{1}{(s+1)^3} - \frac{e^{-2}\,e^{-2s}}{(s+1)^3}.$$

Thus the solution to the initial value problem is

$$y(t) = \mathfrak{L}^{-1}\{Y(s)\} = \mathfrak{L}^{-1}\{\frac{3}{s+1}\} + \mathfrak{L}^{-1}\{\frac{7}{(s+1)^2}\} - \mathfrak{L}^{-1}\{\frac{1}{(s+1)^3}\} - \mathfrak{L}^{-1}\{\frac{e^{-2}\,e^{-2s}}{(s+1)^3}\}.$$

The first of these we can read off our table of transforms: $\mathfrak{L}^{-1}\{\frac{3}{s+1}\} = 3e^{-t}$. The next two follow from the rules for $1/t^n$ and $F(s-a)$: $\mathfrak{L}^{-1}\{\frac{7}{(s+1)^2}\} = 7te^{-t}$, and $\mathfrak{L}^{-1}\{\frac{1}{(s+1)^3}\} = \frac{1}{2}\,t^2\,e^{-t}$. Finally, the last term follows from the rule for the unit step function and the transform we just found: $\mathfrak{L}^{-1}\{\frac{e^{-2}\,e^{-2s}}{(s+1)^3}\} = e^{-2}\frac{1}{2}\,(t-2)^2 e^{-(t-2)}u(t-2) = \frac{1}{2}e^{-2t}u(t-2)$. The solution to the problem is therefore

$$y(t) = 3e^{-t} + 7te^{-t} + \frac{1}{2}\,t^2\,e^{-t} - \frac{1}{2}e^{-t}(t-2)^2 u(t-2).$$

6. [12 points] For each of the following, circle True or False to indicate whether the statement is true or not, and provide a one-sentence explanation. ***Note** that without an explanation no credit will be awarded.*

a. [3 points] Given any two solutions y_1 and y_2 to an equation $y'' + p(x)y' + q(x)y = 0$ (where $p(x)$ and $q(x)$ are continuous), any other solution to the equation may be written as $y = c_1y_1 + c_2y_2$ for some constants c_1 and c_2.

True $\boxed{\text{False}}$

Solution: This is false; if the solutions y_1 and y_2 are not linearly independent their linear combination does not give the general solution to the differential equation.

b. [3 points] $\mathfrak{L}^{-1}\{\frac{3s-6}{s^2+4s+20}\} = 3e^{-2t}\cos(4t) - \frac{3}{2}e^{-2t}\sin(4t)$.

True $\boxed{\text{False}}$

Solution: Note that $\frac{3s-6}{s^2+4s+20} = \frac{3(s+2)-12}{(s+2)^2+16}$, so that the inverse transform should be $3e^{-2t}\cos(4t) - 3e^{-2t}\sin(4t)$.

c. [3 points] Any 2nd or higher order system of ordinary differential equations $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$ $(n \geq 2)$ may be written as a system of first-order ordinary differential equations.

$\boxed{\text{True}}$ False

Solution: Let $x_1 = y$, $x_2 = y'$,$\ldots$, $x_n = y^{(n-1)}$, to get the system $x_1' = x_2$, $x_2' = x_3$,$\ldots$, $x_n' = f(t, x_1, x_2, \ldots, x_n)$.

d. [3 points] Suppose we approximate the solution to $y' = f(x, y)$ on the domain $0 \leq x \leq 10$ using a numerical method. With $h = .1$ we get $y(10) \approx 1.501$; with $h = .01$, $y(10) \approx 1.487$; and with $h = 0.001$ or $h = 0.0001$, $y(10) \approx 1.486$. Thus the exact value of $y(10)$ is to three decimal places 1.486, the errors when $h = 0.1$ and $h = 0.01$ are 0.015 and 0.001 respectively, and it is most likely that the numerical method used was the improved Euler method.

True $\boxed{\text{False}}$

Solution: All of the statements are reasonable except for the assertion that improved Euler was used. We would expect that the error would drop by a factor of $10^2 = 100$ (that is, that the error for $h = 0.01$ would be on the order of 0.00015, not 0.001).

7. [14 points] The van der Pohl oscillator is a circuit that may be modeled with the system of differential equations

$$x' = -y, \qquad y' = x + (a - y^2)\, y,$$

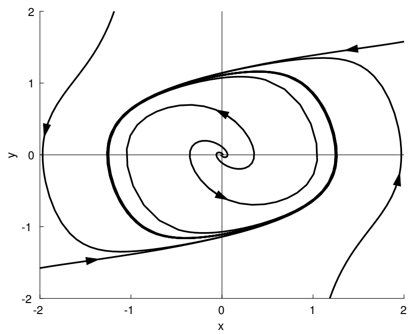
where x is the charge on a capacitor in the circuit and y is current in the circuit, scaled appropriately. The constant a is a parameter in the system.

a. [3 points] Find all critical points for this system.

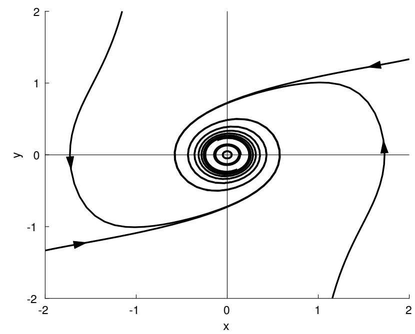
Solution: Setting $x' = y' = 0$, we have $0 = y$ (from the first equation) and $0 = x - (\alpha + x^2)y$ (from the second). With $y = 0$, the second reduces to $x = 0$. Thus the only critical point is $(x, y) = (0, 0)$.

b. [6 points] The two phase portraits (I and II) shown below are generated for the system two of the three cases $a = -1$, $a = 0$ or $a = 1$. By doing a linear analysis of the system at your critical points, determine which cases these match and explain why.

I.



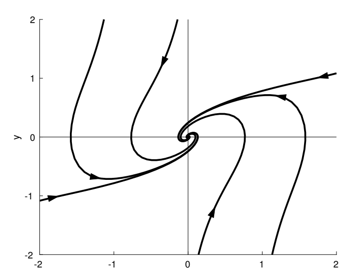
II.



Solution: The Jacobian for the system is $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & a - 3y^2 \end{pmatrix}$, which is at $(0, 0)$ the matrix $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$. The eigenvalues of this matrix are given by $-\lambda(a - \lambda) + 1 = 0$, so that $(\lambda - \frac{a}{2})^2 + 1 - a^2/4 = 0$ and $\lambda = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}$. If $a = -1$, we have $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; if $a = 0$, $\lambda = \pm i$, and if $a = 1$, $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The last ($a = 1$) corresponds to an unstable spiral source, which is what is shown in phase portrait (I). The middle case ($a = 0$) is a center, which seems characteristic of phase portrait (II).

c. [5 points] Based on your linear analysis, sketch a phase portrait for the last of the three cases $a = -1$, $a = 0$, or $a = 1$.

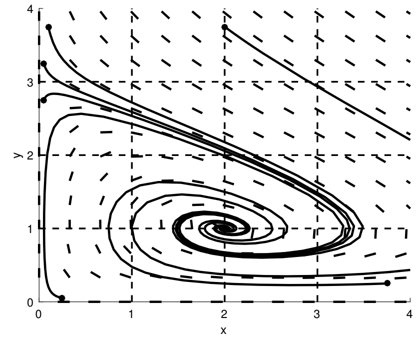
Solution: The remaining case is $a = -1$, a stable spiral sink. Note that if we start at $(x, y) = (1, 0)$, we have $(x', y') = (0, 1)$, so the spirals must be counter-clockwise. This leads to the phase portrait shown below.



8. [12 points] Consider the system

$$x' = ax + 2xy$$
$$y' = 3y - y^2 + bxy$$

where a and b are constants. The direction field and phase portrait for the system are shown in the figure to the right. (Dots indicate initial conditions for the trajectories shown.)



a. [6 points] What are a and b? Be sure to explain your answer.

Solution: From the phase portrait, it appears that there are three critical points, at $(0,0)$, $(0,3)$ and $(2,1)$. The given system will have critical points when

$$0 = x(a + 2y) \qquad \text{and}$$
$$0 = y(3 - y + bx).$$

Thus, from the first equation we have $x = 0$ or $y = -a/2$. If $x = 0$ the second equation requires that $y = 0$ or $y = 3$, which give the first two critical points. Thus $a = -2$ so that $y = 1$ for the third critical point. If $y = 1$ we have $2 + bx = 0$, and we know that $x = 2$. Thus $b = -1$.

b. [6 points] Suppose that x and y are populations of interacting species. What type of interaction is being modeled here? Explain what the phase portrait shown tells you about the behavior and expected long-term values of the populations and sketch a representative solution (x and y) against t.

Solution: With $a = -2$ and $b = -1$, we have the system

$$x' = -2x + 2xy$$
$$y' = 3y - y^2 - xy.$$

Thus, population x is helped by the interaction while population y is hindered; we might guess that this is a predator/prey relationship with x as the predator and y as the prey. From the phase portrait, we see that if we start with $y = 0$ (no prey), $x \to 0$; and if $x = 0$ (no predators), $y \to 3$ (the environmental limiting value). If $x > 0$ and $y > 0$, we expect to see the populations of x and y oscillating around $x = 2$, $y = 1$ and as $t \to \infty$ that we will see $x \to 2$ and $y \to 1$. Thus our sketch of solutions is the following: