Math 216 — First Midterm

6 February, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find real-valued solutions to each of the following, as indicated. If possible, find an explicit expression for y. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] Find the general solution to $y' + 5y = 3e^{6t}$

b. [5 points] Find the solution to $(t+1)y' + y = 3$, $y(0) = 2$.

c. [5 points] Find the general solution to $y' + y^2 = ty^2$.

2. [14 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] The general solution to $x' = x + 8y$, $y' = 2x + y$.

b. [7 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$.

3. [14 points] Lake Michigan has a volume of about 4,900 km^3 of water. Each year about 158 km^3 of that flows out to Lake Huron, and we may assume that an equal amount of water flows in from the rivers feeding the lake, rainfall and snowmelt. (Of course, the loss should really take into account evaporation as well, but ignore that here.)

a. [4 points] Write a differential equation modeling the amount $p(t)$ of a pollutant in the lake, assuming that the pollutant is added at a constant rate I_0 per year.

b. [6 points] For this and part **(c)** suppose that the equation that you obtained in **(a)** is $p' + \frac{1}{20}p = I_0$, and that the rate at which pollutant is added changes at at $t = 4$ as regulations on allowed pollution released are loosened. Thus, instead of a constant I_0, we have $I_0(t) = \begin{cases} 100, & t < 4 \\ 1000, & t \geq 4 \end{cases}$. Find $p(t)$ if $p(0) = 500$. You need not simplify any constants in your answer.

c. [4 points] For the initial value problem you solved in **(b)**, on what domain does the solution exist, and where is it unique? On what domain would we expect a unique solution given our existence and uniqueness theorem? Is our result here consistent with the theorem?

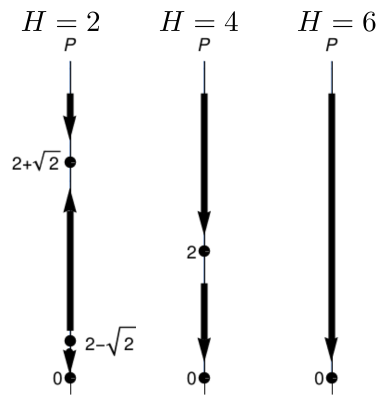
4. [14 points] Consider a population P that is modeled by the first-order differential equation $P' = f(P)$. In this problem we consider only $P \geq 0$, as a negative population is not physically relevant.

a. [4 points] If the phase line for the population is shown to the right, what could the differential equation be? Why?

b. [6 points] Now suppose that $f(P)$ depends on a parameter H, which measures the amount of harvesting of the population (e.g., if the population was fish, H could measure how many of the fish are caught through fishing). If the phase lines for $H = 2$, $H = 4$, and $H = 6$ are shown to the right, which, if any, of the following equations could model the population? Explain.

i. $P' = -P(P-1)(P-H)$ ii. $P' = P^3 - 4P^2 + HP$

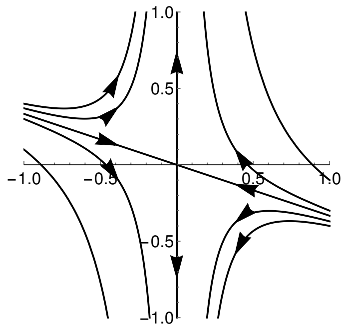
iii. $P' = -P(P^2 - HP + 4)$ iv. $P' = -P(P^2 - 4P + H)$



c. [4 points] Finally, sketch a qualitatively accurate plot of solutions to the equation for the case $H = 4$.

5. [15 points] For each of the following the given figure is a phase portrait for a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant 2×2 matrix. For each select the correct characterization of the eigenvalues of $\mathbf{A}$ and fill in the requested information about an eigenvector of this matrix.

a. [5 points]

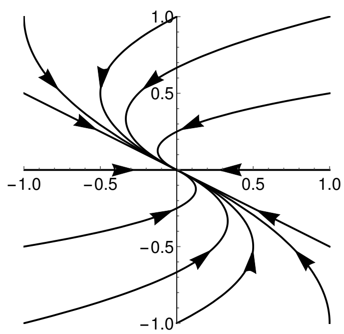


The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2$; $\lambda_1 = -1,\ \lambda_2 = 2$;
$\lambda_1 = -1,\ \lambda_2 = -2$; $\lambda_{1,2} = 1 \pm i$;
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"):

b. [5 points]



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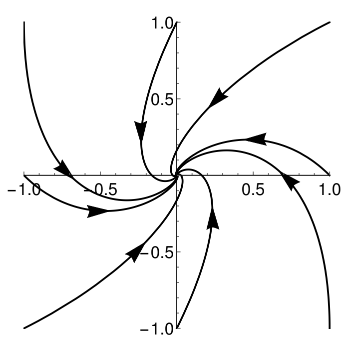
c. [5 points]



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6. [12 points] Identify each of the following as true or false, by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why you selected that answer.

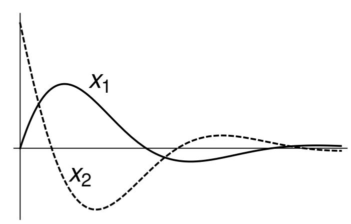
a. [3 points] The initial value problem $(y^2 - 1)y' = (t - 1)$, $y(0) = 0$, is guaranteed to have a unique solution for all times $t > 0$.

True False

b. [3 points] If the eigenvalues of a 2×2 constant, real-valued matrix $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = 1$, then the system of algebraic equations $\mathbf{Ax} = \mathbf{0}$ has infinitely many nonzero solutions.

True False

c. [3 points] If $\mathbf{A} = \begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}$, then component plots for the system of equations $\mathbf{x}' = \mathbf{Ax}$ will appear as in the figure to the right for all real values of a.



True False

d. [3 points] A first-order problem such as $y' = t\sin(y) + \cos(y)$, which is neither linear nor separable, is amenable to qualitative analysis by drawing a phase line and sketching qualitatively accurate solution curves.

True False

7. [16 points] The van der Pol equation has the form $x'' + \mu \frac{df}{dx} x' + x = 0$. In this problem suppose that $f(x) = -\sin(x)$, so that the equation becomes $x'' - \mu \cos(x) x' + x = 0$.

a. [4 points] Letting $x_1 = x$ and $x_2 = x'$, write this as a system of two first-order differential equations in x_1 and x_2.

b. [4 points] Use a Taylor expansion to linearize the original equation at the critical point $x = 0$.

Problem 7, continued.

c. [4 points] Suppose that the equation you obtained in **(b)** is, for some value of μ,

$$x'' + 3x' + 2x = 0.$$

Write this as a matrix equation in $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and solve it.

d. [4 points] Sketch a phase portrait given your solution in **(c)**. What does it tell us about the long-term behavior of the current x in the circuit?

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