Math 216 — Second Midterm

20 March, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Solve $\frac{1}{3}\,y'' + 2y' + 3y = 2t$, $y(0) = 0$, $y'(0) = \frac{4}{3}$.

b. [7 points] Find the general solution to $y'' + 2y' + 5y = 2te^{-t}$.

2. [14 points] Find each of the following, providing an explicit formula where appropriate. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] $Y(s) = \mathcal{L}\{y(t)\}$ if $y'' + 4y' + 20y = 3\sin(2t)$, $y(0) = 1$, $y'(0) = 2$.

b. [5 points] $\mathcal{L}^{-1}\{\frac{s}{s^2+4s+5}\}$

c. [4 points] Using the integral definition of the Laplace transform, derive the transform rule $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$ for a function $f(t)$ with transform $L\{f(t)\} = F(s)$. (Recall $u_c(t)$ is the unit step function at $t = c$, $u_c(t) = \begin{cases} 0, & 0 < t < c \\ 1, & t \geq c \end{cases}$.)

3. [14 points] Use Laplace transforms to solve each of the following.

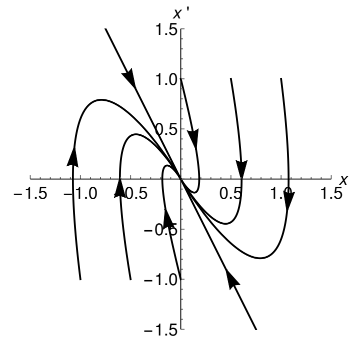
a. [7 points] $y'' + 4y' + 4y = 2e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

b. [7 points] $y'' + 3y' = \begin{cases} 12, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$, $y(0) = 0$, $y'(0) = 0$.

4. [14 points] Consider a mass-spring system modeled by

$$x'' + 4x' + \alpha x = 0.$$

a. [5 points] Suppose that the phase portrait for the system is that shown to the right, below. For what values of α, if any, will the system have this type of behavior? Explain.



b. [3 points] For what values of α, if any, will the system be underdamped? Critically damped? Overdamped? Explain how you obtain your answers.

c. [6 points] Let $\alpha = 6$. How will the phase portrait for the system in this case differ from that given in **(a)**? Sketch the phase portrait for this case. In a separate graph, sketch representative solutions $x(t)$ as functions of time for the case $\alpha = 4$. (Note that you do not need to solve the problem to do this.)

5. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] For the system $x' = -xy + y^2$, $y' = x^2 - xy$, the nonlinear trajectory in the phase plane with $x(0) = -3$ and $y(0) = 0$ lies on a circle centered on the origin.

True False

b. [3 points] For a linear differential operator $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$, if y_1 and y_2 are different functions satisfying $L[y_1] = L[y_2] = g(t) \neq 0$, then, for any constants c_1 and c_2, $y = c_1 y_1 - c_2 y_2$ satisfies $L[y] = 0$.

True False

c. [3 points] The solution to a differential equation $my'' + ky = F(t)$ modeling the motion y of an undamped mechanical spring system with a periodic external force $F(t) = F_0 \cos(\omega t)$ can always be written as $y = A\cos(\omega_0 t - \delta_1) + B\cos(\omega t - \delta_2)$, a sum of two oscillatory terms. (A, B, ω_0, δ_1 and δ_2 are constants.)

True False

d. [3 points] If $\lambda^2 + p\lambda + q = 0$ is the characteristic equation of a constant-coefficient linear differential equation $L[y] = g(t)$, then solving for $Y(s) = \mathcal{L}\{y(t)\}$ will result in an expression involving a product of $(s^2 + ps + q)^{-1}$ with other terms.

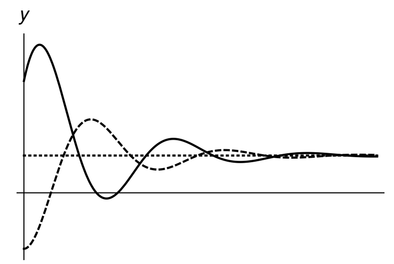
True False

e. [3 points] If $f(t) \neq 0$ has Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t), & 0 < t < c \\ 0, & t \geq c \end{cases}$, then $\mathcal{L}\{g(t)\} = (1 - e^{-sc})F(s)$.

True False

6. [14 points] In the following, we consider the behavior of solutions to a linear, second-order, constant-coefficient differential equation with a forcing term.

a. [5 points] Write a differential equation of this type that could have the three solution curves given to the right. Explain how you know your answer is correct.



b. [6 points] Now suppose that the general solution to the problem is $y = (c_1 + c_2 t + t\ln(t))e^{-t}$. What is the differential equation, including the forcing term? Explain.

c. [3 points] If you were finding, by hand, the general solution given in **(b)**, what method or methods could you use? In these methods, what form do you guess for the solution?

7. [15 points] In lab 3 we considered a nonlinear system modeling a laser with a slightly varying gain rate, which we rewrite slightly in this problem as

$$N' = \gamma(A - N(1+P))$$
$$P' = P(N-1)$$

with $A = A_0 + \epsilon\cos(\omega t)$.

a. [5 points] If A is constant, the system has a critical point $(N, P) = (1, A-1)$. Let $N = 1 + u$, $P = A_0 - 1 + v$, and $A = A_0 + \epsilon\cos(\omega t)$ and find a linear system in u and v by assuming that u, v and ϵ are all very small.

b. [5 points] The system you obtained in **(a)** can be rewritten, for some constants α and β, as $v'' + \alpha v' + \beta v = \epsilon\beta\cos(\omega t)$. Find the steady-state response to this rewritten form.

Problem 7, continued.

c. [5 points] Suppose that the steady-state solution that you obtained in **(b)** was, for some constant b with $|b| < 1$, $v_{ss} = \frac{b\omega}{(1-\omega^2)^2+(b\omega)^2}\cos(\omega t) + \frac{1-\omega^2}{(1-\omega^2)^2+(b\omega)^2}\sin(\omega t)$. Find the amplitude of the oscillation and explain why the solution exhibits resonance behavior.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$