Math 216 — Final Exam

24 April, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the solution to $y' + \sin(t)y = 3\sin(t)$, $y(0) = 2$.

b. [6 points] The general solution to $y'' + 3y' - 4y = 2 - e^t$.

2. [12 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the general solution to $\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

b. [6 points] Find the solution to $y'' + 2y' = \delta(t-1)$, $y(0) = 0$, $y'(0) = 3$.

3. [12 points] Consider a skydiver who jumps from a plane at time $t = 0$. She falls, affected by gravity and air resistance, until at a time $t = t_d$ she deploys her parachute, changing the force of air resistance. Let v be her downward velocity and g be the acceleration due to gravity (9.81 m/s in metric units, 32.2 ft/sec^2 in English).

a. [4 points] Explain why a reasonable model for v is $v' = g - \begin{cases} k_1 v, & t < t_d \\ k_2 v, & t \geq t_d \end{cases}$, $v(0) = 0$.

(Here, k_1 and k_2 are different constants, with $k_2 \gg k_1$.)

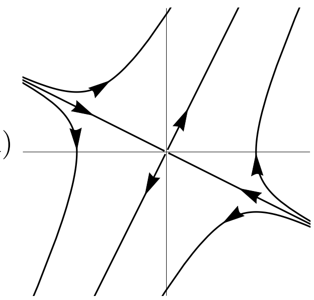
b. [4 points] Rewrite this model as a single equation involving a step function.

c. [4 points] Explain where you would run into difficulty if you tried to use Laplace transforms to solve your equation from (b).

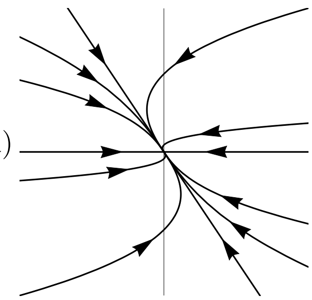
4. [12 points] Consider the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for each of the following matrices $\mathbf{A}$. Assuming also that each of k_1, k_2 and k_3 are positive constants, match each of the following matrices to one of the phase portraits given to the right. Explain how you are able to make this matching.

a. [4 points] $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -(k_1+1) & -2 \end{pmatrix}$

(i)

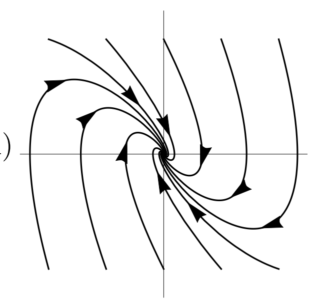


(ii)



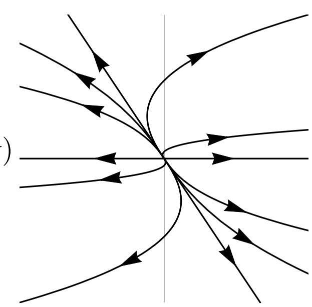
b. [4 points] $\mathbf{A} = \begin{pmatrix} k_2+1 & 1 \\ 0 & 1 \end{pmatrix}$

(iii)



c. [4 points] $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2k_3 \end{pmatrix}$

(iv)



5. [12 points] Consider a mass-spring system in which the displacement of the mass is modeled by the initial value problem

$$y'' + 2ky' + 4y = 5\cos(\omega t), \quad y(0) = y'(0) = 0.$$

a. [6 points] Suppose that the damping is very small, so that we may assume that $k = 0$. Assuming that $\omega \neq 2$, find the displacement y of the mass, in the form $y = R\cos(\omega t - \delta) + C\cos(\omega_0 t - \delta_0)$.

b. [6 points] Next consider the four cases (1) $k = 0$, $\omega = 1.9$; (2) $k = 0$, $\omega = 2$; (3) $k = 0.1$, $\omega = 2$; and (4) $k = 0.1$, $\omega = 10$. Sketch qualitatively accurate graphs of the displacements y as functions of time t for each of these cases, giving some sense of the relative magnitude of the solutions. Briefly explain why you sketch the graphs you do. (Note that you do not need to solve the problem again to answer this question.)

6. [10 points] If we make a small typographical error when writing out the Lorenz system that we studied in lab 5, we obtain the system

$$\begin{aligned} x' &= \sigma(-x+y) \\ y' &= ry - x - xz \\ z' &= -bz + xy \end{aligned}$$

a. [5 points] As with the Lorenz system, one critical point of this system is $(0,0,0)$. Find a linear system that approximates the system near $(0,0,0)$.

b. [5 points] If $b = 5$, $\sigma = 1$, and $r = 1/4$, the eigenvalues and eigenvectors of the coefficient matrix of the linearized system you found in (a) are approximately $\lambda_1 = -5$ and $\lambda_{2,3} = -\frac{3}{8} \pm \frac{7}{9}\,i$, with $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_{2,3} = \begin{pmatrix} \frac{5}{8} \pm \frac{6}{7}\,i \\ 1 \\ 0 \end{pmatrix}$. Describe phase space trajectories in this case. If we start with an initial condition $(x, y, z) = (0.5, 0.5, 0)$, sketch the trajectory in the phase space.

7. [15 points] Consider a mass-spring system with a nonlinear "soft" spring, for which the displacement x of a mass attached to the spring is modeled by

$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$

a. [4 points] Rewrite this as a system in $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$.

b. [5 points] Find all critical points for your system from (a).

Problem 7, continued.
We are solving

$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$

You may want to write your system from (a) here:

c. [6 points] Let $\gamma_0 = 4$ and $k = 18$. Sketch the phase plane for the system in this case by linearizing about all critical points and determining local behavior. Using your sketch, what do you expect to happen to a solution that starts with the initial condition $x(0) = 0.8$, $x'(0) = y(0) = 0.2$? *(Note: for this part of the problem you should assume that the original equation is in fact well-defined for $x < 0$.)*

8. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] Two linearly independent solutions of $x'' + 6x' + 9x = 0$ are $x_1 = e^{-3t}$ and $x_2 = te^{-3t}$. Thus two linearly independent solutions of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \mathbf{x}$ are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t}$.

True False

b. [3 points] If $\mathbf{A}$ is a real-valued 5×5 matrix with 5 distinct eigenvalues, not necessarily real, and if the real parts of all of the eigenvalues are negative, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

True False

c. [3 points] If the nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has an unstable isolated critical point $\mathbf{x} = \mathbf{x}_0$, then any solution to the system will eventually get infinitely far from $\mathbf{x}_0$.

True False

d. [3 points] Suppose that the nonlinear system $x' = F(x, y)$, $y' = G(x, y)$ has an isolated critical point $(x, y) = (1, 2)$. If we are able to linearize the system at this critical point and the eigenvalues of the resulting coefficient matrix are real-valued and non-zero, we can deduce the stability of the critical point from the linearization.

True False

e. [3 points] $\mathcal{L}^{-1}\{\frac{e^{-2s}s}{(s+1)^2+4}\} = e^{-(t-2)}\cos(2(t-2))u_2(t)$

True False

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\frac{s}{s^2 + a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t - c)$	e^{-cs}
A.	$f'(t)$	$s\, F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\, f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s - c)$
D.	$u_c(t)\, f(t - c)$	$e^{-cs}\, F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$