Math 216 — First Midterm

6 February, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find real-valued solutions to each of the following, as indicated. If possible, find an explicit expression for y. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] Find the general solution to $y' + 5y = 3e^{6t}$

Solution: This is first-order and linear. An integrating factor is $\mu(t) = e^{\int 5\,dt} = e^{5t}$, so $(e^{5t}\,y)' = 3e^{11t}$. Integrating both sides, we have $e^{5t}\,y = \frac{3}{11}\,e^{11t} + C$, so that

$$y = \frac{3}{11}\,e^{6t} + C\,e^{-5t}.$$

b. [5 points] Find the solution to $(t+1)y' + y = 3$, $y(0) = 2$.

Solution: Note that this is equivalent to $y' + \frac{1}{t+1}\,y = \frac{3}{t+1}$ and to $y' = -\frac{1}{t+1}(y-3)$, so it is both linear and separable. An integrating factor is $\mu(t) = e^{\int (t+1)^{-1}\,dt} = e^{\ln(t+1)} = t+1$, so, multiplying both sides by μ, we have $((t+1)y)' = 3$. Integrating, $(t+1)y = 3t + C$, so that $y = \frac{3t}{t+1} + \frac{C}{t+1}$. Then, requiring that $y(0) = 2$, we have $C = 2$, and

$$y = \frac{3t}{t+1} + \frac{2}{t+1} = \frac{3t+2}{t+1}.$$

We could also solve this by separating: we have $\frac{y'}{y-3} = -\frac{1}{t+1}$, so that $\ln|y-3| = -\ln|t+1| + c$. Exponentiating both sides and taking $C = \pm e^{c}$, we have $y - 3 = \frac{C}{t+1}$, so that $y = 3 + \frac{C}{t+1}$. With $y(0) = 2$, $C = -1$, so that $y = 3 - \frac{1}{t+1} = \frac{3t+3-1}{t+1} = \frac{3t+2}{t+1}$, as before.

c. [5 points] Find the general solution to $y' + y^2 = ty^2$.

Solution: This is nonlinear, but separable. Separating variables, we have $y^{-2}y' = t - 1$, so that, integrating, $-y^{-1} = \frac{1}{2}t^2 - t + C$, and

$$y = -\frac{1}{\frac{1}{2}t^2 - t + C} = \frac{2}{k + 2t - t^2}.$$

(Where $k = -2C$.)

2. [14 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] The general solution to $x' = x + 8y$, $y' = 2x + y$.

Solution: With $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ this is $\mathbf{x}' = \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} \mathbf{x}$. We then look for $\mathbf{x} = \mathbf{v}e^{\lambda t}$, so that

$$\det\left(\begin{pmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2 - 16 = (\lambda - 1)^2 - 16 = 0.$$

Thus $\lambda - 1 = \pm 4$, and $\lambda = -3, 5$. If $\lambda = -3$, we have $\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. If $\lambda = 5$, $\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0}$, and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t}.$$

b. [7 points] The solution to $\mathbf{x}' = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} -6 \\ 0 \end{pmatrix}$.

Solution: Again looking for $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we have

$$\det\left(\begin{pmatrix} -\lambda & 4 \\ -1 & -\lambda \end{pmatrix}\right) = \lambda^2 + 4 = 0,$$

so that $\lambda = \pm 2i$. If $\lambda = 2i$, we have $\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \mathbf{v} = \mathbf{0}$, so that $\mathbf{v} = \begin{pmatrix} 2 \\ i \end{pmatrix}$. A complex-valued solution is

$$\mathbf{x} = \begin{pmatrix} 2 \\ i \end{pmatrix} (\cos(2t) + i\sin(2t) = \begin{pmatrix} 2\cos(2t) \\ -\sin(2t) \end{pmatrix} + i \begin{pmatrix} 2\sin(2t) \\ \cos(2t) \end{pmatrix} = \mathbf{a} + i\mathbf{b},$$

so a real-valued general solution is $\mathbf{x} = c_1\mathbf{a} + c_2\mathbf{b}$. Applying the initial condition, $c_1 = -3$ and $c_2 = 0$, so that

$$\mathbf{x} = \begin{pmatrix} -6\cos(2t) \\ 3\sin(2t) \end{pmatrix}.$$

3. [14 points] Lake Michigan has a volume of about 4,900 km^3 of water. Each year about 158 km^3 of that flows out to Lake Huron, and we may assume that an equal amount of water flows in from the rivers feeding the lake, rainfall and snowmelt. (Of course, the loss should really take into account evaporation as well, but ignore that here.)

a. [4 points] Write a differential equation modeling the amount $p(t)$ of a pollutant in the lake, assuming that the pollutant is added at a constant rate I_0 per year.

Solution: We have $p' =$ (rate in)$-$(rate out). The rate in is given to be I_0, and, assuming that the lake water is well mixed, the rate out will be (pollutant concentration)(158) $= \frac{p}{4900}(158)$. Thus we have

$$p' = I_0 - \frac{158}{4900}\,p, \quad \text{or} \quad p' + \frac{158}{4900}\,p = I_0.$$

b. [6 points] For this and part **(c)** suppose that the equation that you obtained in **(a)** is $p' + \frac{1}{20}\,p = I_0$, and that the rate at which pollutant is added changes at at $t = 4$ as regulations on allowed pollution released are loosened. Thus, instead of a constant I_0, we have $I_0(t) = \begin{cases} 100, & t < 4 \\ 1000, & t \geq 4 \end{cases}$. Find $p(t)$ if $p(0) = 500$. You need not simplify any constants in your answer.

Solution: For $t < 4$, we solve with the integrating factor $\mu = e^{t/20}$: $(e^{t/20}p)' = 100e^{t/20}$, so $p = 2000 + Ce^{-t/20}$. The initial condition requires that $C = 500 - 2000 = -1500$, so $p = 2000 - 1500e^{-t/20}$. Then, for $t \geq 4$, we have $p' + \frac{1}{20}\,p = 1000$. Proceeding as before, $p = 20{,}000 + Ce^{-t/20}$, with initial condition $p(4) = 2000 - 1500e^{-1/5}$. Thus $2000 - 1500e^{-1/5} = 20{,}000 + Ce^{-1/5}$, and $C = -18{,}000e^{1/5} - 1500$. The solution to the problem is therefore

$$p(t) = \begin{cases} 2{,}000 - 1{,}500e^{-t/20} & t < 4 \\ 20{,}000 - (18{,}000e^{1/5} + 1{,}500)e^{-t/20} & t \geq 4. \end{cases}$$

c. [4 points] For the initial value problem you solved in **(b)**, on what domain does the solution exist, and where is it unique? On what domain would we expect a unique solution given our existence and uniqueness theorem? Is our result here consistent with the theorem?

Solution: We've found a function that is continuous for all t and unique, but it has a discontinuous derivative at $t = 4$. For it to be a solution to the differential equation, it must be differentiable (we've defined a solution to be a differentiable function). Therefore, we have a unique solution on $0 \leq t < 4$. The existence and uniqueness theorem guarantees a unique solution on the interval where the forcing and coefficient functions in the differential equation are continuous, which is $0 \leq t < 4$. Our result is therefore consistent with that (as it, of course, must be).

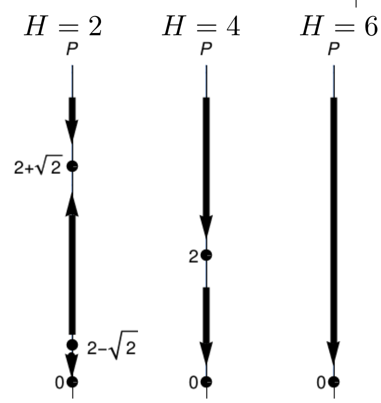
4. [14 points] Consider a population P that is modeled by the first-order differential equation $P' = f(P)$. In this problem we consider only $P \geq 0$, as a negative population is not physically relevant.

a. [4 points] If the phase line for the population is shown to the right, what could the differential equation be? Why?

Solution: There are many possible solutions to this; we need the function $f(P)$ to have zeros at $P = 0$, $P = 1$, and $P = 3$, and to be negative for $0 < P < 1$ and $P > 1$ and positive for $1 < P < 3$. One such function is $f(P) = -P(P-1)(P-3)$, so that $P' = -P(P-1)(P-3)$.

b. [6 points] Now suppose that $f(P)$ depends on a parameter H, which measures the amount of harvesting of the population (e.g., if the population was fish, H could measure how many of the fish are caught through fishing). If the phase lines for $H = 2$, $H = 4$, and $H = 6$ are shown to the right, which, if any, of the following equations could model the population? Explain.

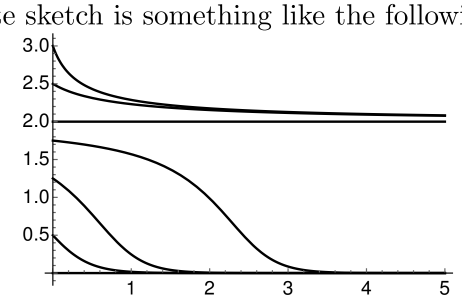
i. $P' = -P(P-1)(P-H)$ ii. $P' = P^3 - 4P^2 + HP$
iii. $P' = -P(P^2 - HP + 4)$ iv. $P' = -P(P^2 - 4P + H)$



Solution: We must have equilibrium solutions as shown, and the derivative must have the appropriate sign to give the indicated phase lines. In particular, for large P we must have $P' < 0$: this disqualifies (ii). Then, when $H = 4$ the roots of the expression on the right-hand side of the equation must be $P = 0, 2$: this disqualifies (i). Finally, note that (iii) has roots $P = 0$ and $P = \frac{1}{2}H \pm \frac{1}{2}\sqrt{H^2 - 16}$. When $H = 6$ this has a positive root, which shouldn't be the case, so it is also not correct. By elimination, the equation must be (iv); this has roots $P = 0$ and $P = 2 \pm \sqrt{4 - H}$, which gives exactly the phase lines shown. This is therefore correct.

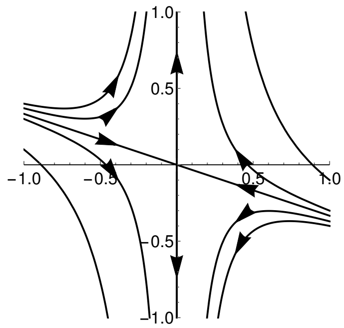
c. [4 points] Finally, sketch a qualitatively accurate plot of solutions to the equation for the case $H = 4$.

Solution: An appropriate sketch is something like the following.



5. [15 points] For each of the following the given figure is a phase portrait for a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a constant 2×2 matrix. For each select the correct characterization of the eigenvalues of $\mathbf{A}$ and fill in the requested information about an eigenvector of this matrix.

a. [5 points]



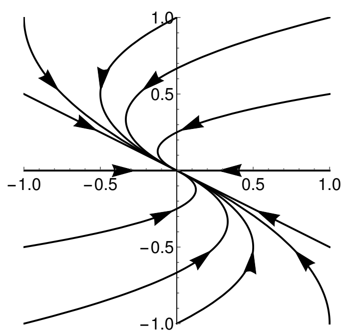
The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2;$ $\boxed{\lambda_1 = -1,\ \lambda_2 = 2;}$
$\lambda_1 = -1,\ \lambda_2 = -2;$ $\lambda_{1,2} = 1 \pm i;$
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solution: There are trajectories approaching and leaving the origin, so we must have a positive and a negative value of λ. Trajectories leave the origin along the y-axis, so one eigenvector must be $\mathbf{v} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$. (The other has trajectories which converge to the origin, and is $\mathbf{v} = \begin{pmatrix} 3 & -1 \end{pmatrix}^T$, but this isn't possible to determine exactly.)

b. [5 points]



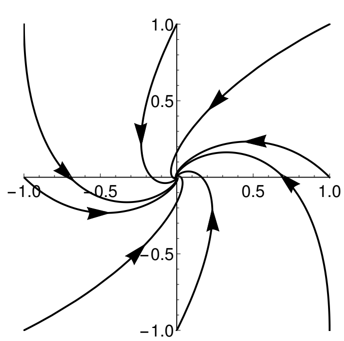
The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2;$ $\lambda_1 = -1,\ \lambda_2 = 2;$
$\boxed{\lambda_1 = -1,\ \lambda_2 = -2;}$ $\lambda_{1,2} = 1 \pm i;$
$\lambda_{1,2} = -1 \pm i$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution: There are two straight line trajectories (one along the x-axis), so the eigenvalues and vectors must be real, and one eigenvector must be $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$. Both eigenvalues are negative because all trajectories approach the origin. (The second eigenvector looks to be, and is, $\mathbf{v} = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$.)

c. [5 points]



The eigenvalues of $\mathbf{A}$ could be (circle one):

$\lambda_1 = 1,\ \lambda_2 = 2;$ $\lambda_1 = -1,\ \lambda_2 = 2;$
$\lambda_1 = -1,\ \lambda_2 = -2;$ $\lambda_{1,2} = 1 \pm i;$
$\boxed{\lambda_{1,2} = -1 \pm i}$

If possible, give one eigenvector of $\mathbf{A}$ (if it is not possible, write "n/a"): **n/a**

Solution: There are no straight line solutions, and it appears that the trajectories all spiral in to the origin, so λ must be complex, and we cannot tell what the eigenvectors are.

6. [12 points] Identify each of the following as true or false, by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why you selected that answer.

a. [3 points] The initial value problem $(y^2 - 1)y' = (t - 1)$, $y(0) = 0$, is guaranteed to have a unique solution for all times $t > 0$.

True $\boxed{\text{False}}$

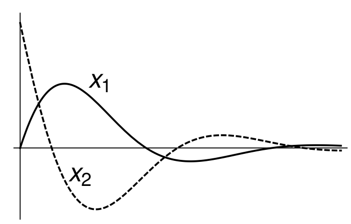
Solution: We note that this is a nonlinear equation of the from $y' = f(t, y)$, with f and $\frac{\partial f}{\partial y}$ being discontinuous only at $y = \pm 1$. Thus we are guaranteed a unique solution through the initial condition, but the interval on which it exists may be constrained. In this case, it exists only for $t < 1 + \sqrt{\frac{7}{3}}$ (though this isn't immediately obvious from the equation).

b. [3 points] If the eigenvalues of a 2×2 constant, real-valued matrix $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = 1$, then the system of algebraic equations $\mathbf{Ax} = \mathbf{0}$ has infinitely many nonzero solutions.

$\boxed{\text{True}}$ False

Solution: If $\lambda = 0$ is an eigenvalue, then we know that $\det(\mathbf{A} - 0\mathbf{I}) = \det(\mathbf{A}) = 0$. This means that $\mathbf{Ax} = \mathbf{b}$ has no or an infinite number of solutions for any $\mathbf{b}$; if $\mathbf{b} = \mathbf{0}$, there must be an infinite number, of which the zero solution is one.

c. [3 points] If $\mathbf{A} = \begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}$, then component plots for the system of equations $\mathbf{x}' = \mathbf{Ax}$ will appear as in the figure to the right for all real values of a.



$\boxed{\text{True}}$ False

Solution: Technically this is false, but if one is not being too tricky and assumes that $a \neq 0$, it is true. In that case eigenvalues of $\mathbf{A}$ are given by $(\lambda+1)^2 = -a^2$, so $\lambda = -1 \pm ia$, and solutions will be decaying and oscillatory, which is what is shown here. If $a = 0$, however, we have $\lambda = -1$, twice, and there are two linearly independent eigenvectors. Thus in that case we would have strictly decaying solutions. (Either of these responses were accepted as correct for this problem.)

d. [3 points] A first-order problem such as $y' = t\sin(y) + \cos(y)$, which is neither linear nor separable, is amenable to qualitative analysis by drawing a phase line and sketching qualitatively accurate solution curves.

True $\boxed{\text{False}}$

Solution: This type of qualitative analysis only works with autonomous equations, for which the dependence on the independent variable t is implicit.

7. [16 points] The van der Pol equation has the form $x'' + \mu \frac{df}{dx} x' + x = 0$. In this problem suppose that $f(x) = -\sin(x)$, so that the equation becomes $x'' - \mu \cos(x) x' + x = 0$.

a. [4 points] Letting $x_1 = x$ and $x_2 = x'$, write this as a system of two first-order differential equations in x_1 and x_2.

Solution: We have

$$x_1' = x_2$$
$$x_2' = -x_1 + \mu \cos(x_1) x_2.$$

b. [4 points] Use a Taylor expansion to linearize the original equation at the critical point $x = 0$.

Solution: Using the Taylor series for $\cos(x)$, we have $\cos(x) = 1 - \frac{1}{2}x^2 + \cdots$, so that the equation is $x'' - \mu(1 - \frac{1}{2}x^2 + \cdots)x' + x = 0$. Expanding and dropping nonlinear terms, we have

$$x'' - \mu x' + x = 0.$$

Problem 7, continued.

c. [4 points] Suppose that the equation you obtained in **(b)** is, for some value of μ,

$$x'' + 3x' + 2x = 0.$$

Write this as a matrix equation in $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and solve it.

Solution: We have

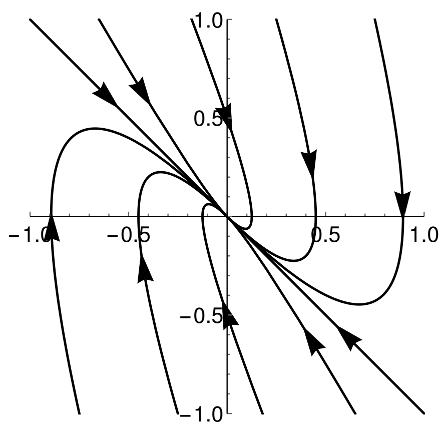
$$\begin{array}{l} x_1' = x_2 \\ x_2' = -2x_1 - 3x_2 \end{array}, \quad \text{or} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Letting $\mathbf{x} = \mathbf{v}e^{\lambda t}$, we must have $\det(\mathbf{A} - \lambda\mathbf{I}) = (-\lambda)(-\lambda - 3) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0$. Thus $\lambda = -2$ and $\lambda = -1$. The eigenvector for $\lambda = -2$ satisfies $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Similarly, for $\lambda = -1$, we have $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0}$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution to the problem is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}.$$

d. [4 points] Sketch a phase portrait given your solution in **(c)**. What does it tell us about the long-term behavior of the current x in the circuit?

Solution: The two straight-line solutions in the problem lie along $y = -x$ and $y = -x/2$, and the latter decays much faster than the former. Thus we have the phase portrait shown below.



This indicates that the current ($x = x_1$) will asymptotically approach 0 for all initial currents.