Math 216 — Second Midterm

20 March, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [14 points] Find real-valued solutions for each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [7 points] Solve $\frac{1}{3}\,y'' + 2y' + 3y = 2t$, $y(0) = 0$, $y'(0) = \frac{4}{3}$.

Solution: The algebra may be easier if we first multiply by 3, obtaining $y''+6y'+9y = 6t$. The characteristic equation for this is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$, so $\lambda = -3$ twice, and the homogeneous solution is $y_c = c_1 e^{-3t} + c_2 te^{-3t}$. To find y_p we use the method of undetermined coefficients, guessing $y_p = At + B$. Then, plugging in,

$$6A + 9At + 9B = 6t,$$

so that $A = \frac{2}{3}$ and $B = -\frac{4}{9}$. The general solution is

$$y = c_1 e^{-3t} + c_2 te^{-3t} + \frac{2}{3}\,t - \frac{4}{9}.$$

Applying the initial conditions, we have $y(0) = c_1 - \frac{4}{9} = 0$, so $c_1 = \frac{4}{9}$, and $y'(0) = -3c_1 + c_2 + \frac{2}{3} = c_2 - \frac{2}{3} = \frac{4}{3}$, so that $c_2 = 2$. Thus

$$y = \frac{4}{9}\,e^{-3t} + 2te^{-3t} + \frac{2}{3}\,t - \frac{4}{9}.$$

b. [7 points] Find the general solution to $y'' + 2y' + 5y = 2te^{-t}$.

Solution: The general solution will be $y = y_c + y_p$, where y_c solves the complementary homogenous problem and y_p is a particular solutions. For y_c we guess $y = e^{\lambda t}$, so that $\lambda^2 + 2\lambda + 5 = (\lambda + 1)^2 + 4 = 0$, and $\lambda = -1 \pm 2i$. Thus $y_c = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. For y_p we use the method of undetermined coefficients, taking $y_p = (At+B)e^{-t}$. Plugging in, we have

$$(At - 2A + B)e^{-t} + 2(-At + A - B)e^{-t} + 5(At + B)e^{-t} = 3te^{-t}.$$

Collecting terms in e^{-t} and te^{-t}, we have $4B = 0$ and $4A = 2$. Thus $B = 0$ and $A = \frac{1}{2}$, and

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{1}{2}\,te^{-t}.$$

2. [14 points] Find each of the following, providing an explicit formula where appropriate. *(Note that minimal partial credit will be given on this problem.)*

a. [5 points] $Y(s) = \mathcal{L}\{y(t)\}$ if $y'' + 4y' + 20y = 3\sin(2t)$, $y(0) = 1$, $y'(0) = 2$.

Solution: Transforming both sides of the equation, we have

$$s^2Y - s - 2 + 4(sY - 1) + 20Y = \frac{6}{s^2+4},$$

so that

$$Y = \frac{s+6}{s^2+4s+20} + \frac{6}{(s^2+4)(s^2+4s+20)}.$$

b. [5 points] $\mathcal{L}^{-1}\{\frac{s}{s^2+4s+5}\}$

Solution: This is

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{s}{s^2+4s+5}\} &= \mathcal{L}^{-1}\{\frac{(s+2)-2}{(s+2)^2+1}\}\\ &= \mathcal{L}^{-1}\{\frac{s+2}{(s+2)^2+1}\} - \mathcal{L}^{-1}\{\frac{2}{(s+2)^2+1}\}\\ &= e^{-2t}\cos(t) - 2e^{-2t}\sin(t).\end{aligned}$$

c. [4 points] Using the integral definition of the Laplace transform, derive the transform rule $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$ for a function $f(t)$ with transform $L\{f(t)\} = F(s)$. (Recall $u_c(t)$ is the unit step function at $t = c$, $u_c(t) = \begin{cases} 0, & 0 < t < c \\ 1, & t \geq c \end{cases}$.)

Solution: The integral definition is $\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)\,dt$. Noting that $u_c(t)$ is zero for $t < c$, we may rewrite this as an integral with lower bound $t = c$. With the substitution $w = t - c$, we have

$$\begin{aligned}\int_0^\infty e^{-st}u_c(t)f(t-c)\,dt = \int_c^\infty e^{-st}f(t-c)\,dt &= \int_0^\infty e^{-s(w+c)}f(w)\,dw\\ &= e^{-cs}\int_0^\infty e^{-sw}f(w)\,dw = e^{-cs}F(s).\end{aligned}$$

3. [14 points] Use Laplace transforms to solve each of the following.

a. [7 points] $y'' + 4y' + 4y = 2e^{-2t}$, $y(0) = 1$, $y'(0) = 0$.

Solution: Taking the Laplace transform of both sides of the equation, we have with $Y = \mathcal{L}\{y\}$,

$$s^2Y - s + 4sY - 4 + 4Y = \frac{2}{s+2}, \quad \text{or} \quad Y = \frac{2}{(s+2)^3} + \frac{s+4}{(s+2)^2}.$$

From transforms 3 and C from the table, we see that the first term in Y will invert as $\mathcal{L}^{-1}\{\frac{2}{(s+2)^3}\} = t^2\,e^{-2t}$. To do the second, we use partial fractions: $\frac{s+4}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$. Clearing the denominators, we have $s+4 = A(s+2)+B$, so that with $s=-2$ we find $B=2$. Then $s=0$ requires that $A=1$, so that $\mathcal{L}^{-1}\{\frac{s+4}{(s+2)^2}\} = \mathcal{L}^{-1}\{\frac{1}{s+2} + \frac{2}{(s+2)^2}\} = e^{-2t} + 2te^{-2t}$. Combining this with the first result, we have

$$y = \mathcal{L}^{-1}\{Y\} = t^2e^{-2t} + e^{-2t} + 2te^{-2t}.$$

b. [7 points] $y'' + 3y' = \begin{cases} 12, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$, $y(0) = 0$, $y'(0) = 0$.

Solution: We want to transform both sides of the equation; the right-hand side we can do by using the definition of the transform, or by noting that the differential equation may be written as $y'' + 3y' = 12 - 12u_2(t)$. Transforming the equation using transform 6 in the table, we have

$$s^2Y + 3sY = \frac{12}{s} - \frac{12e^{-2s}}{s}, \quad \text{so that} \quad Y = \frac{12}{s^2(s+3)} - \frac{12e^{-2s}}{s^2(s+3)}.$$

To find y we need to invert the transform of $\frac{12}{s^2(s+3)}$. We decompose this with partial fractions: $\frac{12}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$. Clearing the denominators, $12 = As(s+3) + B(s+3) + Cs^2$. If $s=0$, $B=4$; if $s=-3$, $C=\frac{4}{3}$. Then, if $s=-2$, $12 = -2A + 4 + \frac{16}{3}$, so that $A = -\frac{4}{3}$. Thus $\mathcal{L}^{-1}\{\frac{12}{s^2(s+3)}\} = \mathcal{L}^{-1}\{-\frac{4}{3s} + \frac{4}{s^2} + \frac{4}{3(s+3)}\} = -\frac{4}{3} + 4t + \frac{4}{3}e^{-3t}$, and, using this and transform 6 from the table, we have

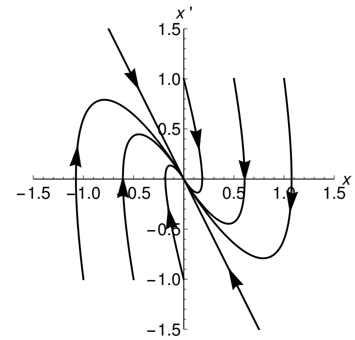
$$y = \mathcal{L}^{-1}\{Y\} = -\frac{4}{3} + 4t + \frac{4}{3}e^{-3t} - \left(-\frac{4}{3} + 4(t-2) + \frac{4}{3}e^{-3(t-2)}\right)u_2(t).$$

4. [14 points] Consider a mass-spring system modeled by

$$x'' + 4x' + \alpha x = 0.$$

a. [5 points] Suppose that the phase portrait for the system is that shown to the right, below. For what values of α, if any, will the system have this type of behavior? Explain.

Solution: The characteristic equation of the equation is $\lambda^2+4\lambda+\alpha = (\lambda+2)^2+\alpha-4 = 0$. The behavior shown in the phase portrait is that of a critically damped system, with a repeated eigenvalue and single eigenvector. This occurs when $\alpha = 4$. Thus we conclude that $\alpha = 4$.

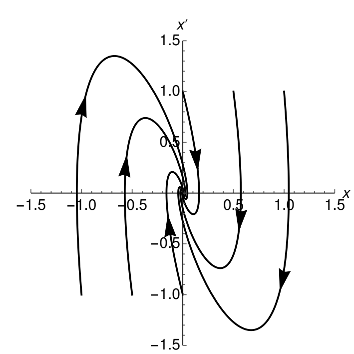


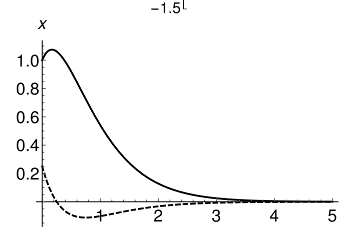
b. [3 points] For what values of α, if any, will the system be underdamped? Critically damped? Overdamped? Explain how you obtain your answers.

Solution: The work from **(a)** shows that for $\alpha = 4$ the system is critically damped. It is underdamped when it has a complex conjugate pair of roots, which will occur when $\alpha > 4$. It is overdamped when $\alpha < 4$.

c. [6 points] Let $\alpha = 6$. How will the phase portrait for the system in this case differ from that given in **(a)**? Sketch the phase portrait for this case. In a separate graph, sketch representative solutions $x(t)$ as functions of time for the case $\alpha = 4$. (Note that you do not need to solve the problem to do this.)

Solution: From the work above, when $\alpha = 6$ the system will be underdamped. We can also see from the critically damped case in part (a) that the critical point $(0, 0)$ will be a sink and that trajectories will spiral in to the origin in a clockwise direction. This is shown in the figure to the right. When $\alpha = 4$ and we have the phase portrait shown in (a), the solution curves must converge to zero, with the possibility of crossing the t-axis (but not repeatedly), as shown in the lower figure.





5. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] For the system $x' = -xy + y^2$, $y' = x^2 - xy$, the nonlinear trajectory in the phase plane with $x(0) = -3$ and $y(0) = 0$ lies on a circle centered on the origin.

$\boxed{\text{True}}$ False

Solution: The system gives $\frac{dy}{dx} = \frac{y'}{x'} = \frac{x(x-y)}{-y(x-y)} = -\frac{x}{y}$. Separating and integrating, we get $y\,dy = -x\,dx$, so that $x^2 + y^2 = 2c$.

b. [3 points] For a linear differential operator $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$, if y_1 and y_2 are different functions satisfying $L[y_1] = L[y_2] = g(t) \neq 0$, then, for any constants c_1 and c_2, $y = c_1y_1 - c_2y_2$ satisfies $L[y] = 0$.

True $\boxed{\text{False}}$

Solution: Relying on the linearity of the operator,

$$L[y] = L[c_1y_1 - c_2y_2] = c_1L[y_1] - c_2L[y_2] = c_1g(t) - c_2g(t) = (c_1 - c_2)g(t),$$

which is zero only if $c_1 - c_2 = 0$.

c. [3 points] The solution to a differential equation $my'' + ky = F(t)$ modeling the motion y of an undamped mechanical spring system with a periodic external force $F(t) = F_0\cos(\omega t)$ can always be written as $y = A\cos(\omega_0 t - \delta_1) + B\cos(\omega t - \delta_2)$, a sum of two oscillatory terms. (A, B, ω_0, δ_1 and δ_2 are constants.)

True $\boxed{\text{False}}$

Solution: This is only true if the forcing frequence ω is not equal to the natural frequency of the system, $\omega_0 = \sqrt{k/m}$. If $\omega = \omega_0$, we will have a growing solution $y = A\cos(\omega_0 t - \delta_1) + Bt\cos(\omega_0 - \delta_2)$.

d. [3 points] If $\lambda^2 + p\lambda + q = 0$ is the characteristic equation of a constant-coefficient linear differential equation $L[y] = g(t)$, then solving for $Y(s) = \mathcal{L}\{y(t)\}$ will result in an expression involving a product of $(s^2 + ps + q)^{-1}$ with other terms.

$\boxed{\text{True}}$ False

Solution: The characteristic equation tells us that the differential equation is $L[y] = y'' + py' + qy = g(t)$. The transform of this is $s^2Y - sy(0) - y'(0) + psY - y(0) + qY = G(s)$, and solving for Y will give the indicated result.

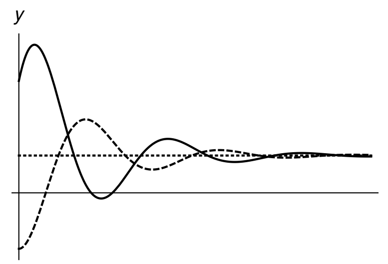
e. [3 points] If $f(t) \neq 0$ has Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t), & 0 < t < c \\ 0, & t \geq c \end{cases}$, then $\mathcal{L}\{g(t)\} = (1 - e^{-sc})F(s)$.

True $\boxed{\text{False}}$

Solution: The easiest way to see this is to look for $h(t) = \mathcal{L}^{-1}\{(1 - e^{-sc})F(s)\} = f(t) - f(t-c)u_c(t)$. Then, by definition, $g(t) = (1 - u_c(t))f(t)$, and in general $h(t) \neq g(t)$.

6. [14 points] In the following, we consider the behavior of solutions to a linear, second-order, constant-coefficient differential equation with a forcing term.

a. [5 points] Write a differential equation of this type that could have the three solution curves given to the right. Explain how you know your answer is correct.



Solution: The constant solution shows that we have a non-zero equilibrium solution, so the forcing term is $g(t) = k$, a constant. Then the two non-constant solutions show an oscillatory transient with decaying amplitude, so the characteristic equation of the differential equation must have complex roots with a negative real part. Thus any equation of the form $y'' + ay' + by = k$, where $a > 0$ and $a^2 - 4b < 0$ (and, because the equilibrium is positive, $k > 0$) will produce the desired result. One such is $y'' + y' + y = 1$.

b. [6 points] Now suppose that the general solution to the problem is $y = (c_1 + c_2 t + t\ln(t))e^{-t}$. What is the differential equation, including the forcing term? Explain.

Solution: Because the problem is linear we know that the general solution has the form $y = y_c + y_p$, where y_c is the solution to the complementary homogeneous problem and y_p is a solution to the problem with forcing. Because it is constant-coefficient and second-order, the solution y_c has terms of the form $e^{\lambda t}$ or $te^{\lambda t}$ (where λ may be zero or complex), so the homogeneous solution here must be $y_c = c_1 e^{-t} + c_2 te^{-t}$. This requires that $\lambda = -1$, twice, so the characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$, and the linear differential operator is $L[y] = y'' + 2y' + y$. Then $y_p = t\ln(t)e^{-t}$. We can find $g(t)$ by plugging this into $L[y]$; to do this, we calculuate $y_p' = -t\ln(t)e^{-t} + (\ln(t) + 1)e^{-t}$ and $y_p'' = -(y_p)' + \frac{1}{t}e^{-t} - (\ln(t) + 1)e^{-t}$. Then

$$\begin{aligned} L[y_p] &= y_p'' + 2y_p' + y_p \\ &= (\frac{1}{t}e^{-t} - (\ln(t) + 1)e^{-t}) + (-t\ln(t)e^{-t} + (\ln(t) + 1)e^{-t}) + t\ln(t)e^{-t} \\ &= t^{-1}e^{-t} = g(t). \end{aligned}$$

Thus the equation and forcing are $y'' + 2y' + y = t^{-1}e^{-t}$.

c. [3 points] If you were finding, by hand, the general solution given in **(b)**, what method or methods could you use? In these methods, what form do you guess for the solution?

Solution: Because the problem is linear, we know that we will be finding the solution to the complementary homogeneous problem and then finding a particular solution. Because it is constant-coefficient, the former will always be done by finding the solution to the eigenvalue problem obtained by looking for solutions of the form $y = e^{\lambda t}$ (or, $\mathbf{x} = \mathbf{v}e^{\lambda t}$ for the equivalent system of two first-order equations). The forcing term (and particular solution) do not admit use of the method of undertermined coefficients, so we would use variation of parameters to guess $y_p = u_1(t)e^{-t} + u_2(t)te^{-t}$.

7. [15 points] In lab 3 we considered a nonlinear system modeling a laser with a slightly varying gain rate, which we rewrite slightly in this problem as

$$\begin{aligned} N' &= \gamma(A - N(1+P)) \\ P' &= P(N-1) \end{aligned}$$

with $A = A_0 + \epsilon\cos(\omega t)$.

a. [5 points] If A is constant, the system has a critical point $(N, P) = (1, A-1)$. Let $N = 1+u$, $P = A_0 - 1 + v$, and $A = A_0 + \epsilon\cos(\omega t)$ and find a linear system in u and v by assuming that u, v and ϵ are all very small.

Solution: Substituting these values into the system, we have

$$\begin{aligned} u' &= \gamma(A_0 + \epsilon\cos(\omega t) - (1+u)(A_0+v)) \\ &= \gamma(-A_0u - v) + \epsilon\gamma\cos(\omega t) - \gamma uv \\ v' &= (A_0 - 1 + v)u = (A_0-1)u + uv \end{aligned}$$

Discarding the (very, very,) very small terms uv, we have

$$u' = \gamma(-A_0u - v) + \epsilon\gamma\cos(\omega t), \quad v' = (A_0-1)u.$$

b. [5 points] The system you obtained in **(a)** can be rewritten, for some constants α and β, as $v'' + \alpha v' + \beta v = \epsilon\beta\cos(\omega t)$. Find the steady-state response to this rewritten form.

Solution: The steady-state response is v_p. Letting $v_p = B\cos(\omega t) + C\sin(\omega t)$ and plugging into the equation, we have

$$\begin{aligned} &-B\omega^2\cos(\omega t) - C\omega^2\sin(\omega t) \\ &\qquad - B\alpha\omega\sin(\omega t) + C\alpha\omega\cos(\omega t) + B\beta\cos(\omega t) + C\beta\sin(\omega t) = \epsilon\beta\cos(\omega t). \end{aligned}$$

Collecting terms in $\cos(\omega t)$ and $\sin(\omega t)$, we have two equations for B and C,

$$\begin{aligned} (-\omega^2 + \beta)B + \alpha\omega C &= \epsilon\beta \\ -\alpha\omega B + (-\omega^2+\beta)C &= 0. \end{aligned}$$

Multiplying the first by $\alpha\omega$ and the second by $-\omega^2 + \beta$ and adding, we find

$$C = \frac{\epsilon\beta\alpha\omega}{(-\omega^2+\beta)^2 + (\alpha\omega)^2}, \quad B = \frac{\epsilon\beta(-\omega^2+\beta)}{(-\omega^2+\beta)^2 + (\alpha\omega)^2},$$

where the result for B follows by solving the second equation for C in terms of B. Then the steady-state response is $v = B\cos(\omega t) + C\sin(\omega t)$ for these B and C.

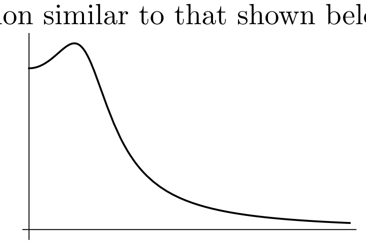
Problem 7, continued.

c. [5 points] Suppose that the steady-state solution that you obtained in **(b)** was, for some constant b with $|b| < 1$, $v_{ss} = \frac{b\omega}{(1-\omega^2)^2+(b\,\omega)^2}\cos(\omega t) + \frac{1-\omega^2}{(1-\omega^2)^2+(b\,\omega)^2}\sin(\omega t)$. Find the amplitude of the oscillation and explain why the solution exhibits resonance behavior.

Solution: Note that this is of the form $v_{ss} = B\cos(\omega t) + C\sin(\omega t) = R\cos(\omega t - \delta)$, with $R = \sqrt{B^2 + C^2}$. Thus the amplitude of the response is

$$R = \sqrt{\left(\frac{b\omega}{(1-\omega^2)^2+(b\,\omega)^2}\right)^2 + \left(\frac{1-\omega^2}{(1-\omega^2)^2+(b\,\omega)^2}\right)^2}$$
$$= \sqrt{\frac{(1-\omega)^2+(b\omega)^2}{((1-\omega^2)^2+(b\,\omega)^2)^2}} = \frac{1}{\sqrt{(1-\omega^2)^2+(b\,\omega)^2}}.$$

We expect this to be a function similar to that shown below



(This can be deduced from the form of R as well: if $\omega = 0$, $R = 1$; if $\omega = 1$, $R = 1/|b| > 1$, and as $\omega \to \infty$, $R \to 0$.) Thus, for some intermediate value of ω (a bit less than one) there is a maximum response amplitude, which is what we mean by resonance.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.

Some Laplace Transforms

	$f(t)$	$F(s)$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\,f(t-c)$	$e^{-cs}\,F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\,dt$