Math 216 — Final Exam

24 April, 2017

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [12 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the solution to $y' + \sin(t)y = 3\sin(t)$, $y(0) = 2$.

Solution: We can solve this with either an integrating factor or by separating variables. An integrating factor is $\mu = \exp(\int \sin(t)\,dt) = \exp(-\cos(t))$. Multiplying the equation by μ, we have $(ye^{-\cos(t)})' = 3\sin(t)e^{-\cos(t)}$. We integrate both sides to get $ye^{-\cos(t)} = 3e^{-\cos(t)} + C$, so that $y = 3 + Ce^{\cos(t)}$. The initial condition gives $y(0) = 3 + Ce^1 = 2$, so $C = -e^{-1}$, and the solution is

$$y = 3 - e^{-1+\cos(t)}.$$

Alternately, separating variables, we have $y' = (3-y)\sin(t)$, so that $y'/(3-y) = \sin(t)$. Integrating both sides, $-\ln|3-y| = -\cos(t) + C'$, so that $3 - y = \tilde{C}e^{\cos(t)}$, and $y = 3 + Ce^{\cos(t)}$, as before. Note that in this case we could also apply the initial condition to find C': $0 = -1 + C'$, so that $C' = 1$, and a(n implicit) solution is $-\ln|3-y| = 1 - \cos(t)$, or

$$\ln|3-y| = \cos(t) - 1.$$

b. [6 points] The general solution to $y'' + 3y' - 4y = 2 - e^t$.

Solution: The general solution will be $y = y_c + y_p$, where y_c and y_p are the complementary homogeneous and particular solutions. We look for $y_c = e^{\lambda t}$, so that $\lambda^2 + 3\lambda - 4 = (\lambda+4)(\lambda-1) = 0$, and $\lambda = -4$ or $\lambda = 1$. Thus $y_c = c_1e^{-4t} + c_2e^t$.
To find y_p, we guess $y_p = A + Bte^t$, multiplying the guess for the e^t portion of the forcing by t because this appears in the homogeneous solution. Noting that $(te^t)' = e^t + te^t$ and $(te^t)'' = 2e^t + te^t$, we have after plugging in, $A = -\frac{1}{2}$ and $B(2e^t + te^t + 3e^t + 3te^t - 4te^t) = B(5e^t) = -e^t$. Thus $B = -\frac{1}{5}$, and our general solution is

$$y = c_1e^{-4t} + c_2e^t - \frac{1}{2} - \frac{1}{5}\,te^t.$$

We could use variation of parameters, but it's not the easiest choice here. We guess $y_p = u_1e^{-4t} + u_2e^t$, so that $u_1'e^{-4t} + u_2'e^t = 0$ and $-4u_1'e^{-4t} + u_2'e^t = 2 - e^t$. Combining these, $-5u_1'e^{-4t} = 2 - e^t$, so $u_1' = -\frac{2}{5}e^{4t} + \frac{1}{5}e^{5t}$ and $u_1 = -\frac{1}{10}e^{4t} + \frac{1}{25}e^{5t}$. Then $u_2' = -u_1'e^{-5t} = \frac{2}{5}e^{-t} - \frac{1}{5}$, so $u_2 = -\frac{2}{5}e^{-t} - \frac{1}{5}t$. Thus

$$y_p = -\frac{1}{10} + \frac{1}{25}e^t - \frac{2}{5} - \frac{1}{5}te^t = -\frac{1}{2} - \frac{1}{5}te^t + \frac{1}{25}e^t,$$

and the last term will combine with the homogeneous solution.

2. [12 points] Find real-valued solutions to each of the following, as indicated. *(Note that minimal partial credit will be given on this problem.)*

a. [6 points] Find the general solution to $\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Solution: We look for $\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p$. The complementary homogeneous solution will be $\mathbf{x}_c = \mathbf{v}e^{\lambda t}$, where λ and $\mathbf{v}$ are the eigenvalues and eigenvectors of the coefficient matrix. The eigenvalues satisfy $\det(\begin{pmatrix} -1-\lambda & 3 \\ 2 & -2-\lambda \end{pmatrix}) = (\lambda+1)(\lambda+2) - 6 = \lambda^2 + 3\lambda - 4 = 0$. Thus, by factoring, the quadratic formula, or the work from problem (1b), $\lambda = -4$ and $\lambda = 1$. If $\lambda = -4$, the components of $\mathbf{v}$ satisfy $3(v_1 + v_2) = 2(v_1 + v_2) = 0$, so that $\mathbf{v}_{-4} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If $\lambda = 1$, $-2v_1 + 3v_2 = 0$, so $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Thus $\mathbf{x}_c = c_1\mathbf{v}_{-4}e^{-4t} + c_2\mathbf{v}_1 e^t$. To find $\mathbf{x}_p$, guess $\mathbf{x}_p = \mathbf{a}$, a constant. Then $-a_1 + 3a_2 + 2 = 0$ and $2a_1 - 2a_2 = 0$. From the second, $a_1 = a_2$, and the first requires $a_2 = a_1 = -1$. Combining this with $\mathbf{x}_c$, we have

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^t - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

b. [6 points] Find the solution to $y'' + 2y' = \delta(t-1)$, $y(0) = 0$, $y'(0) = 3$.

Solution: Because of the impulse forcing we use Laplace transforms. Transforming both sides, we have, with $Y = \mathcal{L}\{y\}$,

$$\mathcal{L}\{y'' + 2y'\} = s^2Y - 3 + 2sY = \mathcal{L}\{\delta(t-1)\} = e^{-s}.$$

Thus $Y = \frac{3}{s(s+2)} + \frac{e^{-s}}{s(s+2)}$. To invert, note that $\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$ if $1 = A(s+2) + Bs$. With $s = 0$, $A = \frac{1}{2}$, and with $s = -2$, $B = -\frac{1}{2}$. Thus we may calculate the inverse transform as

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\frac{3}{s(s+2)} + \frac{e^{-s}}{s(s+2)}\} \\ &= \mathcal{L}^{-1}\{\frac{1}{2}(3 + e^{-s})\left(\frac{1}{s} - \frac{1}{s+2}\right) \\ &= \frac{3}{2}(1 - e^{-2t}) + \frac{1}{2}(1 - e^{-2(t-1)})u_1(t). \end{aligned}$$

3. [12 points] Consider a skydiver who jumps from a plane at time $t = 0$. She falls, affected by gravity and air resistance, until at a time $t = t_d$ she deploys her parachute, changing the force of air resistance. Let v be her downward velocity and g be the acceleration due to gravity (9.81 m/s in metric units, 32.2 ft/sec^2 in English).

a. [4 points] Explain why a reasonable model for v is $v' = g - \begin{cases} k_1 v, & t < t_d \\ k_2 v, & t \geq t_d \end{cases}$, $v(0) = 0$. (Here, k_1 and k_2 are different constants, with $k_2 \gg k_1$.)

Solution: We are applying Newton's law, mass $\cdot$ acceleration $= \sum$(applied forces). Here mass $\cdot$ acceleration $= mv'$, and the applied forces are the force of gravity (mg) and air resistance. Assuming that the air resistance is proportional to the skydiver's velocity, it will be $-\alpha v$, with the negative sign indicating that it works in the direction opposite to her velocity. Thus the sum of the forces will be $mg - \alpha v$. When the parachute deploys the constant α will change from one value α_1 to a larger value α_2. Dividing both sides by m and letting $k_{1,2} = \alpha_{1,2}/m$, we have the given model. The initial condition $v(0) = 0$ indicates that the skydiver has no initial downward velocity.

b. [4 points] Rewrite this model as a single equation involving a step function.

Solution: We have

$$v' = 10 - 0.01v(1 - u_{10}(t)) - v\,u_{10}(t) = 10 - 0.01v - 9.99\,v\,u_{10}(t),$$

still with $v(0) = 0$.

c. [4 points] Explain where you would run into difficulty if you tried to use Laplace transforms to solve your equation from (b).

Solution: Let $V = \mathcal{L}\{v\}$. Then, taking the Laplace transform of both sides of the equation, we have

$$s^2 V = \frac{10}{s} - 0.01V - 9.99\mathcal{L}\{v(t)\,u_{10}(t)\}.$$

The difficulty is in calculating $\mathcal{L}\{v(t)\,u_{10}(t)\}$: transform (D) from the table gives $\mathcal{L}\{v(t-10)\,u_{10}(t)\}$, but that is not what we have here. If we attempt to find the transform directly, we have

$$\mathcal{L}\{v(t)\,u_{10}(t)\} = \int_0^\infty v(t)\,u_{10}(t)\,e^{-st}\,dt = \int_{10}^\infty v(t)e^{-st}\,dt,$$

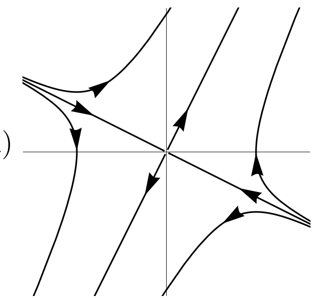
but we have no easy way to relate this to the transform of V: $\int_{10}^\infty v(t)e^{-st}\,dt = \int_0^\infty v(t+10)e^{-10s}e^{-st}\,dt = e^{-10s}\int_0^\infty v(t+10)e^{-st}\,dt$, which is still not tractable.

4. [12 points] Consider the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for each of the following matrices $\mathbf{A}$. Assuming also that each of k_1, k_2 and k_3 are positive constants, match each of the following matrices to one of the phase portraits given to the right. Explain how you are able to make this matching.

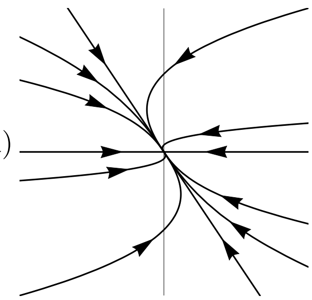
a. [4 points] $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -(k_1+1) & -2 \end{pmatrix}$

Solution: For this matrix we see that eigenvalues are given by the characteristic equation $-\lambda(-2-\lambda)+k_1+1 = \lambda^2+2\lambda+k_1+1 = (\lambda+1)^2+k_1 = 0$. Thus $\lambda = -1 \pm i\sqrt{k}_1$, and, because $k_1 > 0$ we know that the eigenvalues are complex, with negative real part. The phase portrait must therefore show an inward spiral, which is the case in phase portrait (iii).

(i)



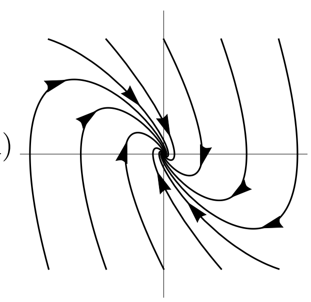
(ii)



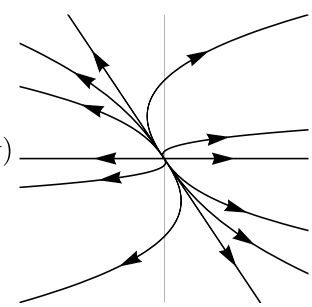
b. [4 points] $\mathbf{A} = \begin{pmatrix} k_2+1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution: For this matrix the characteristic equation is $((k_2+1)-\lambda)(1-\lambda) = 0$, so that the eigenvalues are $\lambda = k_2+1$ and $\lambda = 1$, two distinct positive values. The result will be a node with outward (and two distinct straight-line) trajectories, which is shown in phase portrait (iv).

(iii)



(iv)



c. [4 points] $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2k_3 \end{pmatrix}$

Solution: Finally, for this matrix, we have $-\lambda(2k_3 - \lambda) - 1 = \lambda^2 - 2k_3\lambda - 1 = (\lambda - k_3)^2 - 1 - k_3^2 = 0$. Thus $\lambda = k_3 \pm \sqrt{1+k_3^2}$, and, with $k_3 > 0$, we will have one positive and one negative real eigenvalue. This will give a saddle point, which is shown in phase portrait (i).

5. [12 points] Consider a mass-spring system in which the displacement of the mass is modeled by the initial value problem

$$y'' + 2ky' + 4y = 5\cos(\omega t), \quad y(0) = y'(0) = 0.$$

a. [6 points] Suppose that the damping is very small, so that we may assume that $k = 0$. Assuming that $\omega \neq 2$, find the displacement y of the mass, in the form $y = R\cos(\omega t - \delta) + C\cos(\omega_0 t - \delta_0)$.

Solution: The homogeneous solution is $y = c_1\cos(2t) + c_2\sin(2t)$. To find the particular solution we guess $y_p = A\cos(\omega t) + B\sin(\omega t)$. Plugging in, we have

$$-\omega^2(A\cos(\omega t) + B\sin(\omega t)) + 4(A\cos(\omega t) + B\sin(\omega t)) = 5\cos(\omega t),$$

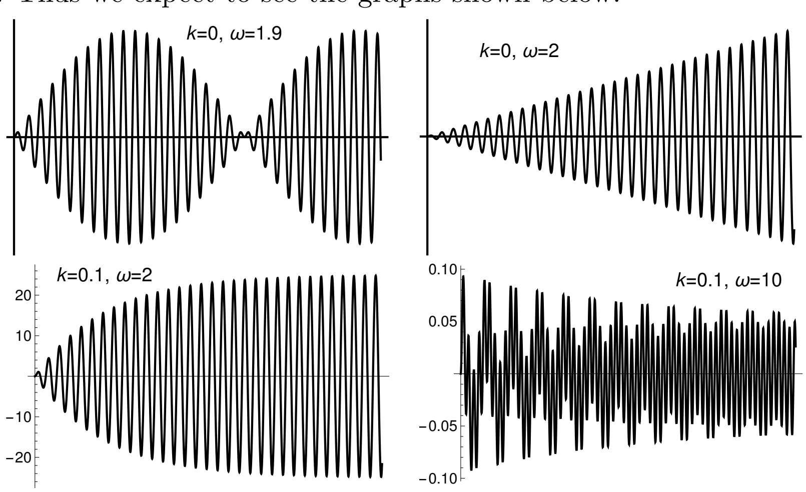
so that $B = 0$ and $A = 5/(4-\omega^2)$. The general solution is therefore $y = c_1\cos(2t) + c_2\sin(2t) + \frac{5}{4-\omega^2}\cos(\omega t)$. The initial conditions require that $c_1 = -\frac{5}{4-\omega^2}$ and $c_2 = 0$, so our solution is in the desired form,

$$y = \frac{5}{4-\omega^2}\cos(\omega t) - \frac{5}{4-\omega^2}\cos(2t),$$

with $R = -C = \frac{5}{4-\omega^2}$ and $\delta = \delta_0 = 0$.

b. [6 points] Next consider the four cases (1) $k = 0$, $\omega = 1.9$; (2) $k = 0$, $\omega = 2$; (3) $k = 0.1$, $\omega = 2$; and (4) $k = 0.1$, $\omega = 10$. Sketch qualitatively accurate graphs of the displacements y as functions of time t for each of these cases, giving some sense of the relative magnitude of the solutions. Briefly explain why you sketch the graphs you do. (Note that you do not need to solve the problem again to answer this question.)

Solution: The first two cases are undamped; as demonstrated by the undefined magnitude R in (a) for $\omega = 2$, we can get pure resonance in this case, in which the magnitude of the solution grows (linearly). When $\omega = 1.9$, near this resonant frequency, we expect to see beats, an oscillatory solution with a slowly modulated large amplitude. When $k > 0$ all solutions are bounded, the homogeneous component of the solution dies away exponentially, and as ω gets very large we expect the amplitude of the steady state solution to get small. Thus we expect to see the graphs shown below.



6. [10 points] If we make a small typographical error when writing out the Lorenz system that we studied in lab 5, we obtain the system

$$\begin{aligned} x' &= \sigma(-x+y) \\ y' &= ry - x - xz \\ z' &= -bz + xy \end{aligned}$$

a. [5 points] As with the Lorenz system, one critical point of this system is $(0,0,0)$. Find a linear system that approximates the system near $(0,0,0)$.

Solution: We know that this will be $\mathbf{x}' = \mathbf{J}(0,0,0)\,\mathbf{x}$, where $\mathbf{J}$ is the Jacobian. This is

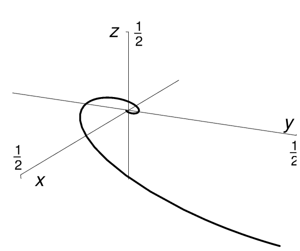
$$\begin{aligned} \mathbf{J}_0 &= \begin{pmatrix} \frac{\partial}{\partial x}\sigma(-x+y) & \frac{\partial}{\partial y}\sigma(-x+y) & \frac{\partial}{\partial z}\sigma(-x+y) \\ \frac{\partial}{\partial x}(ry-x-xz) & \frac{\partial}{\partial y}(ry-x-xz) & \frac{\partial}{\partial z}(ry-x-xz) \\ \frac{\partial}{\partial x}(-bz+xy) & \frac{\partial}{\partial y}(-bz+xy) & \frac{\partial}{\partial z}(-bz+xy) \end{pmatrix}\Bigg|_{(0,0,0)} \\ &= \begin{pmatrix} -\sigma & \sigma & 0 \\ -1-z & r & -x \\ y & x & -b \end{pmatrix}\Bigg|_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ -1 & r & 0 \\ 0 & 0 & -b \end{pmatrix}. \end{aligned}$$

And our system is, with $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$, $\mathbf{x}' = \mathbf{J}_0\,\mathbf{x}$.

Alternately, because we are linearizing at $(0,0,0)$ and all terms are polynomial, we can just drop the nonlinear terms from the system, obtaining the same result.

b. [5 points] If $b = 5$, $\sigma = 1$, and $r = 1/4$, the eigenvalues and eigenvectors of the coefficient matrix of the linearized system you found in (a) are approximately $\lambda_1 = -5$ and $\lambda_{2,3} = -\frac{3}{8} \pm \frac{7}{9}\,i$, with $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_{2,3} = \begin{pmatrix} \frac{5}{8} \pm \frac{6}{7}\,i \\ 1 \\ 0 \end{pmatrix}$. Describe phase space trajectories in this case. If we start with an initial condition $(x,y,z) = (0.5, 0.5, 0)$, sketch the trajectory in the phase space.

Solution: We note that the first eigenvalue is a much larger (in magnitude) negative real value than the real part of the other two; thus, we expect that to decay much faster than the other. Because this is associated with the eigenvector $\mathbf{v}_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, this means that the trajectories will rapidly decay into the xy-plane. Once there, they will be spiral trajectories. We note that the linearized system gives $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -1 & 1 \\ -1 & 1/4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so starting from $(x,y) = (1,1)$, $(x', y') = (0, -3/4)$ and the trajectory must be moving clockwise. Thus we have a clockwise spiral in the xy-plane, shown below.



7. [15 points] Consider a mass-spring system with a nonlinear “soft” spring, for which the displacement x of a mass attached to the spring is modeled by

$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$

a. [4 points] Rewrite this as a system in $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x' \end{pmatrix}$.

Solution: We have

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ k(x^2 - x) - 2\gamma_0 y \end{pmatrix}.$$

b. [5 points] Find all critical points for your system from (a).

Solution: To find critical points we require that $x' = y' = 0$. Thus $y = 0$, and $x^2 - x = 0$, so $x = 0$ or $x = 1$. The two critical points are $(0, 0)$ and $(1, 0)$.

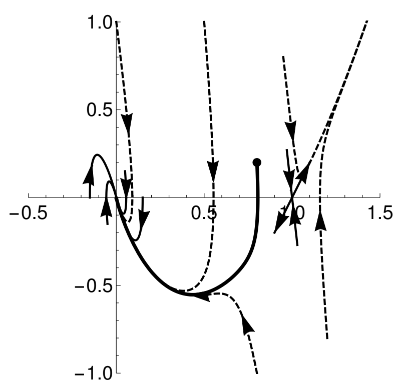
Problem 7, continued.
We are solving
$$x'' + 2\gamma_0 x' + k(x - x^2) = 0.$$
You may want to write your system from (a) here:
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ k(x^2 - x) - 2\gamma_0 y \end{pmatrix}.$$

c. [6 points] Let $\gamma_0 = 4$ and $k = 18$. Sketch the phase plane for the system in this case by linearizing about all critical points and determining local behavior. Using your sketch, what do you expect to happen to a solution that starts with the initial condition $x(0) = 0.8$, $x'(0) = y(0) = 0.2$? *(Note: for this part of the problem you should assume that the original equation is in fact well-defined for $x < 0$.)*

Solution: The Jacobian of the system is $J = \begin{pmatrix} 0 & 1 \\ 18(2x-1) & -8 \end{pmatrix}$. At $(0,0)$, we have $J_0 = \begin{pmatrix} 0 & 1 \\ -18 & -8 \end{pmatrix}$. Eigenvalues of J_0 satisfy $\lambda^2 + 8\lambda + 18 = 0$, or $(\lambda+4)^2 + 2 = 0$, so $\lambda = -4 \pm i\sqrt{2}$. At $(1,0)$, $\begin{pmatrix} x \\ y \end{pmatrix}' = J_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \end{pmatrix}$. This is downward, so near $(0,0)$ we must have a clockwise spiral sink.

At $(1,0)$, the Jacobian is $J_1 = \begin{pmatrix} 0 & 1 \\ 18 & -8 \end{pmatrix}$. Eigenvalues satisfy $\lambda^2 + 8\lambda - 18 = (\lambda+4)^2 - 34 = 0$, so $\lambda = -4 \pm \sqrt{34}$. Note that $\sqrt{34} \approx \sqrt{36} = 6$, so these are approximately $\lambda_{1,2} = -10, 2$. Then eigenvectors satisfy $(4 \mp \sqrt{34})v_1 + v_2 = 0$, so the eigenvectors are $\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ -4 \pm \sqrt{34} \end{pmatrix} \approx \begin{pmatrix} 1 \\ -10 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This is a(n unstable) saddle point. Sketching these in the phase plane, we obtain the graph shown below.

Then, if we start at $(0.8, 0.2)$, we are to the left of the critical point $(1,0)$ and below the attracting eigenline. We therefore expect the trajectory to move out and down toward $(1,0)$, crossing the x-axis to the left of $(1,0)$ and then spiraling in to $(0,0)$. The phase portrait is shown below, with the trajectory starting at $(0.8, 0.2)$ shown with an initial dot and thicker lines. The dashed curves are nonlinear trajectories.



8. [15 points] For each of the following, identify the statement as true or false by circling "True" or "False" as appropriate, and provide a short (one or two sentence) explanation indicating why that answer is correct.

a. [3 points] Two linearly independent solutions of $x'' + 6x' + 9x = 0$ are $x_1 = e^{-3t}$ and $x_2 = te^{-3t}$. Thus two linearly independent solutions of $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \mathbf{x}$ are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} te^{-3t}$.

True $\boxed{\text{False}}$

Solution: There are a number of ways to see that this is false: first is that $\mathbf{x}_2$ isn't a solution to the system ($\mathbf{x}_2' = \begin{pmatrix} -3t+1 \\ 9t-3 \end{pmatrix} e^{-3t}$, while the right hand side is $\begin{pmatrix} -3t \\ 9t \end{pmatrix} e^{-3t}$.) In addition, $\mathbf{x}_1$ and $\mathbf{x}_2$ aren't linearly independent ($W(\mathbf{x}_1, \mathbf{x}_2) = 0$).

b. [3 points] If $\mathbf{A}$ is a real-valued 5×5 matrix with 5 distinct eigenvalues, not necessarily real, and if the real parts of all of the eigenvalues are negative, then $\mathbf{x} = \mathbf{0}$ is an asymptotically stable critical point of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

$\boxed{\text{True}}$ False

Solution: All terms in the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ will be multiplied by a factor of $e^{\mathrm{Re}(\lambda_j)t}$ ($1 \le j \le 5$), with the only other functional dependence being $\cos(\omega t)$, $\sin(\omega t)$ or powers of t. Thus the solutions all decay to zero, and the critical point is stable.

c. [3 points] If the nonlinear system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has an unstable isolated critical point $\mathbf{x} = \mathbf{x}_0$, then any solution to the system will eventually get infinitely far from $\mathbf{x}_0$.

True $\boxed{\text{False}}$

Solution: There are lots of counterexamples. A saddle point has a line of solutions that approach the critical point asymptotically. Another example is the van der Pol system: for certain parameter values $\mathbf{x} = \mathbf{0}$ is an unstable critical point, but all solutions are attracted to a limit cycle which does not allow trajectories to escape to infinity.

d. [3 points] Suppose that the nonlinear system $x' = F(x, y)$, $y' = G(x, y)$ has an isolated critical point $(x, y) = (1, 2)$. If we are able to linearize the system at this critical point and the eigenvalues of the resulting coefficient matrix are real-valued and non-zero, we can deduce the stability of the critical point from the linearization.

$\boxed{\text{True}}$ False

Solution: This is the substance of our theorem about the linear analysis of almost linear systems; if the eigenvalues are real and non-zero, small changes will not change our conclusion as to the stability of the point. If the eigenvalues are equal we may not be able to determine if the point is a node or a spiral point, but its stability will remain the same.

e. [3 points] $\mathcal{L}^{-1}\{\frac{e^{-2s}s}{(s+1)^2+4}\} = e^{-(t-2)}\cos(2(t-2))u_2(t)$

True $\boxed{\text{False}}$

Solution: To use the rule for $F(s-c)$, all terms with s must first be rewritten as $s-c$: $\mathcal{L}^{-1}\{\frac{e^{-2s}s}{(s+1)^2+4}\} = \mathcal{L}^{-1}\{\frac{e^{-2s}((s+1)-1)}{(s+1)^2+4}\} = e^{-(t-2)}(\cos(2(t-2)) - \frac{1}{2}\sin(2(t-2)))u_2(t)$.