Math 216 — First Midterm

13 February, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find real-valued solutions to each of the following, as indicated. Where possible, find explicit solutions.

a. [8 points] $\ln(1/y^2)\,\frac{dy}{dt} = r\,y,\ y(0) = e.$

b. [7 points] Find the general solution to $t\frac{dy}{dt} = ry + t^2$.

2. [15 points] Lake Huron and Lake Erie are two of the Great Lakes, as shown to the right. The volume of Lake Huron is very approximately 4,000 km^3, and that of Lake Erie approximately 500 km^3. We assume that the flow into and out of both lakes is the same, approximately 200 km^3/year, and that all water that flows out of Lake Huron flows into Lake Erie. Suppose that a ruptured oil line adds 30×10^9 kg/year of oil into Lake Huron.

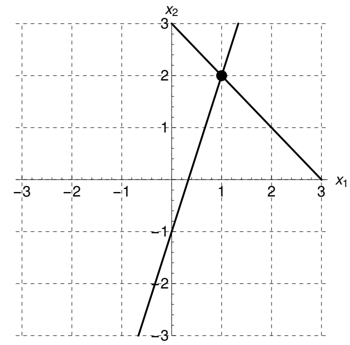
a. [5 points] Write a system of equations for x_1, the amount of oil in Lake Huron, and x_2, the amount in Lake Erie. Assume that the oil is well mixed in either lake, and that the water entering Lake Huron is clean.

b. [5 points] Solve your equation for x_1 directly and use that to solve for x_2. It will likely be convenient to write your answer in terms of the constant $r = 3 \times 10^{10}$.

c. [5 points] Using your work in (a) and (b), write your system from (a) as a matrix equation, and write the solution as a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. What are the eigenvectors of the coefficient matrix in your system?

3. [14 points] In the following, the matrices **A** and **B** are 2×2 real-valued matrices. The vector $\mathbf{x}$ is a 2×1 vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

a. [7 points] If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the solution to $\mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is illustrated in the figure to the right, what are the eigenvalues of **A**?



b. [7 points] Suppose that $\mathbf{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{B}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. What is the general solution to $\mathbf{x}' = \mathbf{B}\mathbf{x}$?

4. [15 points] Find explicit, real-valued solutions to the following, as indicated.

a. [8 points] $x' = x + 2y$, $y' = 3x - 4y$, $x(0) = 1$, $y(0) = 4$.

b. [7 points] $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -8 & -3 \end{pmatrix} \mathbf{x}$.

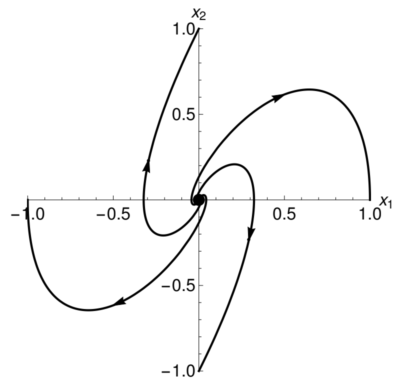
5. [15 points] Consider the initial value problem $y' = -\frac{1}{2}\,y + \sin(y)$, $y(t_0) = y_0$.

a. [5 points] Without trying to solve it, does this initial value problem have a solution? Does your answer depend on the values of t_0 and y_0? Explain.

b. [5 points] By using the Taylor expansion for $\sin(y)$ near the critical point $y = 0$, write a linear equation approximating this equation and solve it. If we start with $y(0) = y_0$ with y_0 small, what does it predict will happen to the solution of the (nonlinear) problem? Is the critical point $y = 0$ stable or unstable?

c. [5 points] Retain another term in the expansion for $\sin(y)$ and write a new differential equation that approximates the equation we started with. Find all critical points, draw a phase line, and explain what it predicts for the behavior of the system for large times.

6. [14 points] Suppose that the phase portrait to the right is the phase portrait for a system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 constant, real-valued matrix. If the system is obtained by rewriting a second order equation as a system of first-order equations, give a possible matrix for $\mathbf{A}$. Explain how you know your choice is correct.



7. [12 points] Suppose that the matrix $\mathbf{A}$ has eigenvalues $\lambda = -1$ and $\lambda = -2$, with corresponding eigenvectors $\mathbf{v}_{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_{-2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If the solution to $\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, sketch the phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Explain how you get your answer.

This page provided for additional work.