Math 216 — Second Midterm

19 March, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, use Laplace transforms, **not** some other solution technique.

a. [8 points] $y'' + 2y' + y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

b. [7 points] $y'' + 6y' + 13y = 0$, $y(0) = 1$, $y'(0) = 0$.

2. [14 points] Fill in the missing portions of each of the following transforms. Briefly explain how you obtain your work.

a. [7 points] $\mathcal{L}\{\begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 5 \\ e^{-(t-5)}, & t \ge 5 \end{cases}\} = \frac{1}{s}(e^{-s} - e^{-5s}) +$ ____________________________

b. [7 points] $\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s^2+1)}\} = 1 - \frac{1}{2}\cos(t) +$ ____________________________

3. [14 points] Find each of the following, as indicated.

a. [7 points] If a function $f(t)$ has the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to find the transform $\mathcal{L}\{\int_0^t f(x)\,dx\}$ in terms of $F(s)$. *(You may assume that $\int_0^\infty f(x)\,dx = L$, a finite value.)*

b. [7 points] Find an explicit expression for $Y = \mathcal{L}\{y\}$ if $y''' + 3y = t^2e^{-4t} - e^{-2t}\cos(5t)$. *(Note that you are not asked to solve the differential equation.)*

4. [15 points] Find explicit, real-valued solutions for each of the following, as indicated. Do **not** use Laplace transform techniques on this problem.

a. [8 points] Find the general solution to $y'' + 2y' + 4y = e^{-t} + t^2$.

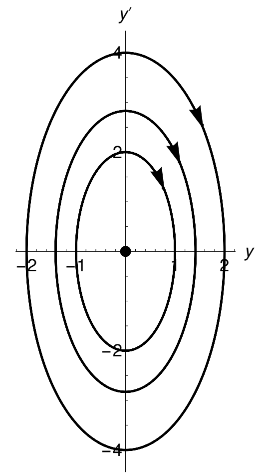
b. [7 points] Solve $y'' + 5y' + 4y = 3\cos(2t)$, $y(0) = 0$, $y'(0) = 1$

5. [14 points] Consider the operators $T[y] = yy'' + 2y^2y'$ and $U[y] = t^2y'' - ty' - 3y$.

a. [9 points] Show that T is nonlinear while U is linear.

b. [5 points] Show that $y_1 = t^{-1}$ and $y_2 = t^3$ constitute a fundamental set of solutions to the equation $U[y] = 0$. What is the general solution to $U[y] = 0$?
(You may assume that $t > 0$.)

6. [13 points] Consider the phase portrait shown to the right, which shows the phase portrait for a linear, second-order, constant coefficient, homogeneous differential equation $L[y] = 0$.



a. [7 points] Write a differential equation that could give this phase portrait. Explain how you obtain your solution, and why is it correct.

b. [6 points] Suppose that we add a forcing term $f(t) = \cos(15t/8)$ to the equation, so that we are solving $L[y] = f(t)$. Sketch an approximate solution curve with $y(0) = 0$, $y'(0) = 1$. Explain why your solution appears as it does.

7. [15 points] In our lab on lasers, we considered a linearization of the nonlinear model for the population inversion N and light intensity P. A critical point of the nonlinear system is $(N, P) = (1, A - 1)$, and linearizing the system near this gives the linear system

$$u' = -\gamma(Au + v), \qquad v' = (A - 1)u,$$

where γ and A are constants.

a. [5 points] Rewrite this as a single, second-order equation in v.

b. [5 points] Suppose that for some α and β your equation from (a) is $v'' + \alpha v' + \beta v = 0$. Under what conditions on α and β will the solution for v be underdamped? Write down two real-valued linearly independent solutions to the equation in this case.

c. [5 points] Now suppose that we force the underdamped equation given in (b) with the periodic forcing term $f(t) = \cos(\omega t)$. Sketch a graph of the steady state solution of the problem. Explain why your graph has the form it does. If ω changes from very small to very large values, how would you expect your sketch to change? Explain.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$;
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^nF(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$