This sample exam is provided to serve as one component of your studying for this exam in this course. Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, use Laplace transforms, **not** some other solution technique.

   a. [8 points] $y'' + 2y' + y = e^{-t}, \ y(0) = 0, \ y'(0) = 1.$

   b. [7 points] $y'' + 6y' + 13y = 0, \ y(0) = 1, \ y'(0) = 0.$
2. [14 points] Fill in the missing portions of each of the following transforms. Briefly explain how you obtain your work.

\begin{align*}
\text{a. [7 points]} & \quad \mathcal{L}\{0, \quad 0 \leq t < 1 \} = \frac{1}{s}(e^{-s} - e^{-5s}) + \underline{\hphantom{\text{expression}}} \\
& \quad \mathcal{L}\{1, \quad 1 \leq t < 5 \} = \underline{\hphantom{\text{expression}}} \\
& \quad \mathcal{L}\{e^{-(t-5)}, \quad t \geq 5 \} = \underline{\hphantom{\text{expression}}} \\
\end{align*}

\begin{align*}
\text{b. [7 points]} & \quad \mathcal{L}^{-1}\left\{\frac{1}{s(s + 1)(s^2 + 1)}\right\} = 1 - \frac{1}{2}\cos(t) + \underline{\hphantom{\text{expression}}} \\
\end{align*}
3. [14 points] Find each of the following, as indicated.

a. [7 points] If a function \( f(t) \) has the Laplace transform \( F(s) = \mathcal{L}\{f(t)\} \), use the integral definition of the Laplace transform to find the transform \( \mathcal{L}\{\int_0^t f(x) \, dx\} \) in terms of \( F(s) \).

(You may assume that \( \int_0^\infty f(x) \, dx = L, \) a finite value.)

b. [7 points] Find an explicit expression for \( Y = \mathcal{L}\{y\} \) if \( y'''' + 3y = t^2 e^{-4t} - e^{-2t} \cos(5t) \).

(Note that you are not asked to solve the differential equation.)
4. [15 points] Find explicit, real-valued solutions for each of the following, as indicated. Do not use Laplace transform techniques on this problem.

a. [8 points] Find the general solution to \( y'' + 2y' + 4y = e^{-t} + t^2 \).

b. [7 points] Solve \( y'' + 5y' + 4y = 3 \cos(2t) \), \( y(0) = 0 \), \( y'(0) = 1 \).
5. [14 points] Consider the operators $T[y] = yy'' + 2y^2y'$ and $U[y] = t^2y'' - ty' - 3y$.

a. [9 points] Show that $T$ is nonlinear while $U$ is linear.

b. [5 points] Show that $y_1 = t^{-1}$ and $y_2 = t^3$ constitute a fundamental set of solutions to the equation $U[y] = 0$. What is the general solution to $U[y] = 0$? (You may assume that $t > 0$.)
6. [13 points] Consider the phase portrait shown to the right, which shows the phase portrait for a linear, second-order, constant coefficient, homogeneous differential equation $L[y] = 0$.

a. [7 points] Write a differential equation that could give this phase portrait. Explain how you obtain your solution, and why is it correct.

b. [6 points] Suppose that we add a forcing term $f(t) = \cos(15t/8)$ to the equation, so that we are solving $L[y] = f(t)$. Sketch an approximate solution curve with $y(0) = 0$, $y'(0) = 1$. Explain why your solution appears as it does.
7. [15 points] In our lab on lasers, we considered a linearization of the nonlinear model for the population inversion $N$ and light intensity $P$. A critical point of the nonlinear system is $(N, P) = (1, A - 1)$, and linearizing the system near this gives the linear system

$$u' = -\gamma(Au + v), \quad v' = (A - 1)u,$$

where $\gamma$ and $A$ are constants.

a. [5 points] Rewrite this as a single, second-order equation in $v$.

b. [5 points] Suppose that for some $\alpha$ and $\beta$ your equation from (a) is $v'' + \alpha v' + \beta v = 0$. Under what conditions on $\alpha$ and $\beta$ will the solution for $v$ be underdamped? Write down two real-valued linearly independent solutions to the equation in this case.

c. [5 points] Now suppose that we force the underdamped equation given in (b) with the periodic forcing term $f(t) = \cos(\omega t)$. Sketch a graph of the steady state solution of the problem. Explain why your graph has the form it does. If $\omega$ changes from very small to very large values, how would you expect your sketch to change? Explain.
This page provided for additional work.
Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  \[
  e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} ;
  \]
  \[
  \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} ;
  \]
  \[
  \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} .
  \]
  
  About $x = 1$: \[
  \ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n} .
  \]

- Some integration formulas: \[
  \int uv' \, dt = uv - \int u'v \, dt ; \text{ thus } \int t e^t \, dt = t e^t - e^t + C , \int t \cos(t) \, dt = t \sin(t) + \cos(t) + C , \text{ and } \int t \sin(t) \, dt = -t \cos(t) + \sin(t) + C .
  \]

- Euler's formula: \[
  e^{i \theta} = \cos \theta + i \sin \theta .
  \]

Some Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t) = \mathcal{L}^{-1}{F(s)}$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
</tr>
</thead>
</table>
| 1. $1$                         | \[
  F(s) = \frac{1}{s} , s > 0
  \]
| 2. $e^{at}$                    | \[
  \frac{1}{s - a} , s > a\]
| 3. $t^n$                       | \[
  \frac{n!}{s^{n+1}}
  \]
| 4. $\sin(at)$                  | \[
  \frac{a}{s^2 + a^2}\]
| 5. $\cos(at)$                  | \[
  \frac{s}{s^2 + a^2}\]
| A. $f'(t)$                     | $s \, F(s) - f(0)$ |
| A.1 $f''(t)$                   | $s^2 \, F(s) - s \, f(0) - f'(0)$ |
| A.2 $f^{(n)}(t)$               | $s^n \, F(s) - \cdots - f^{(n-1)}(0)$ |
| B. $t^n f(t)$                  | $(-1)^n F^{(n)}(s)$ |
| C. $e^{ct} f(t)$               | $F(s - c)$ |