Math 216 — Final Exam

19 April, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions for each of the following, as indicated.

a. [6 points] Solve for y if $y' + ye^t - e^t = 0$, $y(0) = 0$.

b. [9 points] Find the general solution to $\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \mathbf{x}$.

2. [15 points] Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the general solution to $y''' + 4y'' + 3y' = 5 - e^{-2t}$.

b. [8 points] Solve $y'' + 3y' + 2y = 4u_1(t) - 3\delta(t-2)$, $y(0) = 0$, $y'(0) = 1$.

3. [15 points] Consider the differential equation $V' = V^{1/3}(k - V^{2/3})$, where $V(t)$ is some (real-valued) physical quantity and k is a constant.

a. [5 points] Find all equilibrium solutions of the equation and their stability. How does the number of equilibrium solutions depend on k?

b. [5 points] Sketch representative solution curves for the equation. Note that you may need more than one graph if you found in (a) a different number of equilibrium solutions depending on the values of k. In the long run, what solution(s) to the equation do you expect to see?

Problem 3, continued. We are considering the differential equation $V' = V^{1/3}(k - V^{2/3})$*, where* $V(t)$ *is some (real-valued) physical quantity and* k *is a constant.*

c. [5 points] Are there any initial conditions $V(t_0) = V_0$ for which you might expect this differential equation could have no solution? More than one solution? Explain. *(Hint: you shouldn't need to solve the equation to answer this question.)*

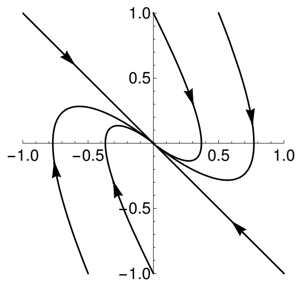
4. [15 points] If the solution to the initial value problem $y'' + 4y' + ay = 3\delta(t - \pi)$, $y(0) = 0$, $y'(0) = k$, is for $t < \pi$ a decaying sinusoid with a local maximum at $t = \pi/2$, and is zero for all values of $t \geq \pi$, what are k and a?

Use Laplace transforms in your solution to this problem.

5. [12 points] Each of the following requires a short (one equation or formula) answer. Provide the required answer, and a short (one or two sentence) explanation.

a. [3 points] Write a linear, constant coefficient, second order, nonhomogeneous differential equation for which the method of undetermined coefficients is not applicable.

b. [3 points] Write a linear, constant coefficient, second order differential equation that has the phase portrait shown to the right.



c. [3 points] If $L[y] = f(t)$ is a linear, constant coefficient, second order differential equation and $L[y] = 0$ is solved by $y = c_1e^{-t} + c_2te^{-t}$, write a function $f(t)$ for which a good solution guess would be $y = At^3e^{-t} + Bt^2e^{-t}$.

d. [3 points] Write a linear, constant coefficient, second order differential equation having a phase portrait that is a spiral sink converging on the point $(2, 0)$.

6. [12 points] Consider the system given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} -3 & 2 & 3 \\ 0 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$. Eigenvalues of $\mathbf{A}$ are $\lambda = -4$, -3, and 2, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, respectively.

a. [4 points] Give an initial condition for which trajectories converge to the origin. Explain how you know your answer is correct.

b. [4 points] Give all initial conditions for which the resulting trajectories remain bounded for all t. Explain.

c. [4 points] Suppose that $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Describe the solution trajectory in phase space. What does it look like as $t \to \infty$? Explain.

7. [16 points] Our model for a ruby laser is, with N = the population inversion of atoms and P = the intensity of the laser,

$$N' = \gamma A - \gamma N(1 + P), \quad P' = P(N - 1).$$

In lab we found that the critical points of this system are $(N, P) = (A, 0)$ and $(N, P) = (1, A - 1)$. For this problem we will assume that $\gamma = \frac{1}{2}$; A is, of course, also a constant.

a. [4 points] Find a linear system that approximates the nonlinear system near the critical point $(A, 0)$. Show that if $A < 1$ this critical point is asymptotically stable, and if $A > 1$ it is unstable.

b. [6 points] Suppose that the linear system you obtained in (a) is, for some value of A, $u' = -\frac{1}{2}\,u - v$, $v' = v$. Sketch a phase portrait that shows solution trajectories of the linear system. Explain how these trajectories are related to trajectories in the (N, P) phase plane.

Problem 7, cont. We are considering the system

$$N' = \frac{1}{2}A - \frac{1}{2}N(1+P), \quad P' = P(N-1),$$

which has critical points $(N,P) = (A,0)$ *and* $(N,P) = (1, A-1)$.

c. [6 points] Suppose that, for the value of A used in (b), the coefficient matrix for the linear system approximating (N,P) near the critical point $(1, A-1)$ is $\begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$, which has eigenvalues $\lambda = \frac{1}{2}(-1 \pm i)$. Using this information with your work in (b), sketch a representative solution curve for P as a function of t, if $P(0) = 0.01$ when $N(0) = 0$.

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Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. The series for $\ln(x)$, taken about $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\, v'\, dt = u\, v - \int u'\, v\, dt$; thus $\int t\, e^t\, dt = t\, e^t - e^t + C$, $\int t\, \cos(t)\, dt = t\, \sin(t) + \cos(t) + C$, and $\int t\, \sin(t)\, dt = -t\, \cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2 + a^2}$
5.	$\cos(at)$	$\frac{s}{s^2 + a^2}$
6.	$u_c(t)$	$\frac{e^{-cs}}{s}$
7.	$\delta(t-c)$	e^{-cs}
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{c\,t} f(t)$	$F(s-c)$
D.	$u_c(t)\, f(t-c)$	$e^{-cs}\, F(s)$
E.	$f(t)$ (periodic with period T)	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dt$
F.	$\int_0^t f(x) g(t-x)\, dx$	$F(s)G(s)$