Math 216 — First Midterm

13 February, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find real-valued solutions to each of the following, as indicated. Where possible, find explicit solutions.

a. [8 points] $\ln(1/y^2)\,\frac{dy}{dt} = r\,y$, $y(0) = e$.

Solution: Separating variables and using rules of logarithms, we have

$$(-2\ln(y))\frac{dy}{y} = r\,dt,$$

so that on integrating both sides we have

$$-(\ln(y))^2 = r\,t + C.$$

Evaluating the initial condition, we have $-1 = C$. We therefore have the implicit solution

$$(\ln(y))^2 = 1 - r\,t.$$

Solving for $\ln(y)$, $\ln(y) = \pm\sqrt{1-rt}$, and we take the positive square root to ensure the initial condition is satisfied. Thus

$$y = e^{\sqrt{1-rt}}.$$

b. [7 points] Find the general solution to $t\frac{dy}{dt} = ry + t^2$.

Solution: Note that this is linear but not separable. Putting it in standard form, we have $\frac{dy}{dt} - \frac{r}{t}\,y = t$, so that an integrating factor is

$$\mu = e^{\int -\frac{r}{t}\,dt} = e^{-r\ln|t|} = t^{-r}.$$

Multiplying by μ, we have $(y\,t^{-r})' = t^{-r+1}$, so that $y\,t^{-r} = \frac{1}{-r+2}t^{-r+2} + C$, and

$$y = \frac{1}{-r+2}\,t^2 + C\,t^r.$$

2. [15 points] Lake Huron and Lake Erie are two of the Great Lakes, as shown to the right. The volume of Lake Huron is very approximately 4,000 km^3, and that of Lake Erie approximately 500 km^3. We assume that the flow into and out of both lakes is the same, approximately 200 km^3/year, and that all water that flows out of Lake Huron flows into Lake Erie. Suppose that a ruptured oil line adds 30×10^9 kg/year of oil into Lake Huron.

a. [5 points] Write a system of equations for x_1, the amount of oil in Lake Huron, and x_2, the amount in Lake Erie. Assume that the oil is well mixed in either lake, and that the water entering Lake Huron is clean.

Solution: For both lakes we have $\frac{dx}{dt}$ = input − output. For Lake Huron the input is $r = 3 \times 10^{10}$ kg/year. The output for both lakes is (200 km^3/year)(concentration), where concentration = x/volume. The output from Lake Huron is the input for Lake Erie. Thus we have the system

$$\frac{dx_1}{dt} = r - \frac{200}{4000}\,x_1, \qquad \frac{dx_2}{dt} = \frac{200}{4000}\,x_1 - \frac{200}{500}\,x_2.$$

b. [5 points] Solve your equation for x_1 directly and use that to solve for x_2. It will likely be convenient to write your answer in terms of the constant $r = 3 \times 10^{10}$.

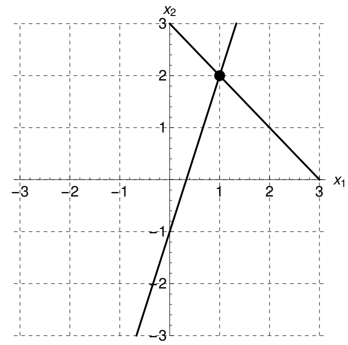
Solution: The equation for x_1 is $x_1' + 0.05x_1 = r$, so an integrating factor is $\mu = e^{0.05t}$. Multiplying by μ and integrating, we have $e^{0.05t}\,x_1 = 20re^{0.05t} + C$, so that $x_1 = 20r + Ce^{-0.05t}$. We assume that $x_1(0) = 0$, so that $x_1 = 20r(1 - e^{-0.05t})$.
Plugging this into the equation for x_2, we have $x_2' + 0.4x_2 = r(1 - e^{-0.05t})$. Proceeding similarly, an integrating factor is $\mu = e^{0.4t}$, so that $(e^{0.4t}\,x_2)' = r\,e^{0.4t} - r\,e^{0.35t}$. Integrating both sides, $e^{0.4t}\,x_2 = \frac{5}{2}\,r\,e^{0.4t} - \frac{20}{7}\,r\,e^{0.35t} + C$, so that $x_2 = \frac{5}{2}\,r - \frac{20}{7}\,r\,e^{-0.05t} + Ce^{-0.4t}$. With $x_2(0) = 0$, $C = (\frac{20}{7} - \frac{5}{2})r = \frac{5}{14}\,r$, and $x_2 = 5r(\frac{1}{2} - \frac{4}{7}\,e^{-0.05t} + \frac{1}{14}\,e^{-0.4t})$.

c. [5 points] Using your work in (a) and (b), write your system from (a) as a matrix equation, and write the solution as a vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. What are the eigenvectors of the coefficient matrix in your system?

Solution: We have $\mathbf{x}' = \begin{pmatrix} -0.05 & 0 \\ 0.05 & -0.4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} r \\ 0 \end{pmatrix}$. The solution is $\mathbf{x} = \begin{pmatrix} -20r \\ -20r/7 \end{pmatrix} e^{-0.05t} + \begin{pmatrix} 0 \\ 5r/14 \end{pmatrix} e^{-0.4t} + \begin{pmatrix} 20r \\ 5r/2 \end{pmatrix}$, from which we see that the eigenvectors are $\begin{pmatrix} 7 \\ 20 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

3. [14 points] In the following, the matrices $\mathbf{A}$ and $\mathbf{B}$ are 2×2 real-valued matrices. The vector $\mathbf{x}$ is a 2×1 vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

a. [7 points] If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and the solution to $\mathbf{A}\mathbf{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ is illustrated in the figure to the right, what are the eigenvalues of $\mathbf{A}$?



Solution: We see that the two lines shown are $x_2 = 3 - x_1$ and $x_2 = 3x_1 - 1$. Because the right-hand side of the system of equations given is $\begin{pmatrix} 3 & -1 \end{pmatrix}$, the equations must be $x_1 + x_2 = 3$ and $-3x_1 + x_2 = -1$. Therefore the rows of $\mathbf{A}$ must be $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} -3 & 1 \end{pmatrix}$, and because the solution shown is $(1, 2)$, the matrix $\mathbf{A}$ must be $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$. The eigenvalues are determined by $(1 - \lambda)(1 - \lambda) + 3 = (\lambda - 1)^2 + 3 = 0$, so that $\lambda = 1 \pm i\sqrt{3}$.

b. [7 points] Suppose that $\mathbf{B}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{B}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$. What is the general solution to $\mathbf{x}' = \mathbf{B}\mathbf{x}$?

Solution: These two systems tell us that the eigenvalues and eigenvectors of $\mathbf{B}$ are $\lambda = 3$ and $\lambda = -2$, with $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. The general solution is therefore $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$.

4. [15 points] Find explicit, real-valued solutions to the following, as indicated.

a. [8 points] $x' = x + 2y$, $y' = 3x - 4y$, $x(0) = 1$, $y(0) = 4$.

Solution: The coefficient matrix for this system is $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$, so that eigenvalues satisfy $(1-\lambda)(-4-\lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda+5)(\lambda-2) = 0$. Thus $\lambda = -5$ or $\lambda = 2$. If $\lambda = -5$, eigenvectors satisfy $6v_1 + 2v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. If $\lambda = 2$, eigenvectors satisfy $-v_1 + 2v_2 = 0$, so that $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

Applying the initial conditions, we have $c_1 + 2c_2 = 1$, and $-3c_1 + c_2 = 4$. Adding three times the first to the second, we have $7c_2 = 7$, so $c_2 = 1$. Then $c_1 = 1 - 2 = -1$, so our solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-5t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

b. [7 points] $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -8 & -3 \end{pmatrix} \mathbf{x}$.

Solution: Eigenvalues of the coefficient matrix are given by $(1-\lambda)(-3-\lambda) + 8 = \lambda^2 + 2\lambda + 5 = (\lambda+1)^2 + 4 = 0$, so $\lambda = -1 \pm 2i$. With $\lambda = -1 + 2i$, the eigenvector satisfies $(2-2i)v_1 + v_2 = 0$, so we have $\mathbf{v} = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix}$ (or any constant multiple thereof; in particular, $\begin{pmatrix} -1-i \\ 4 \end{pmatrix}$ is another option). To find a real-valued solution, we separate out the real and imaginary parts of $\mathbf{x} = \mathbf{v}e^{\lambda t}$. These are the real and imaginary parts of

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix} e^{-t}(\cos(2t) + i\sin(2t)),$$

which are $\mathrm{Re}(\mathbf{x}) = \begin{pmatrix} \cos(2t) \\ -2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}$ and $\mathrm{Im}(\mathbf{x}) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}$. The general solution is therefore

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(2t) \\ -2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) - 2\sin(2t) \end{pmatrix} e^{-t}.$$

(With $\mathbf{v} = \begin{pmatrix} -1-i \\ 4 \end{pmatrix}$, we get $\mathbf{x} = c_1 \begin{pmatrix} -\cos(2t) + \sin(2t) \\ 4\cos(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\cos(2t) - \sin(2t) \\ 4\sin(2t) \end{pmatrix} e^{-t}$.)

5. [15 points] Consider the initial value problem $y' = -\frac{1}{2}\,y + \sin(y)$, $y(t_0) = y_0$.

a. [5 points] Without trying to solve it, does this initial value problem have a solution? Does your answer depend on the values of t_0 and y_0? Explain.

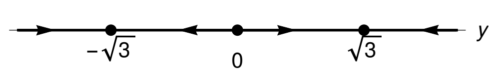
Solution: We see that $f(t, y) = -\frac{1}{2}\,y + \sin(y)$ is continuous and has a continuous first partial f_y (for all t and y). Therefore, our theorem on the existence and uniqueness of solutions guarantees that there is a unique solution through $y(t_0) = y_0$ for any t_0 and any y_0. However, we do not, a priori, know the full domain of this solution.

b. [5 points] By using the Taylor expansion for $\sin(y)$ near the critical point $y = 0$, write a linear equation approximating this equation and solve it. If we start with $y(0) = y_0$ with y_0 small, what does it predict will happen to the solution of the (nonlinear) problem? Is the critical point $y = 0$ stable or unstable?

Solution: We have $\sin(y) \approx y - \frac{1}{6}y^3 + \frac{1}{120}y^5 - \cdots$, so our linear approximation is $y' = \frac{1}{2}\,y$. The solution is tremendously easy: $y = y_0 e^{t/2}$, and solutions grow. Thus, this predicts that for the nonlinear problem, if we start with y_0 close to zero solution trajectories will grow away from $y = y_0$. The critical point is unstable.

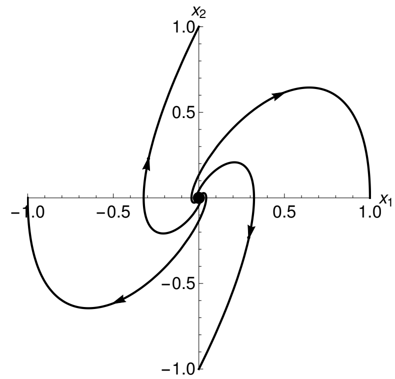
c. [5 points] Retain another term in the expansion for $\sin(y)$ and write a new differential equation that approximates the equation we started with. Find all critical points, draw a phase line, and explain what it predicts for the behavior of the system for large times.

Solution: If we retain another term, we have $y' = \frac{1}{2}\,y - \frac{1}{6}\,y^3$. Critical points are given by $0 = y(1 - \frac{1}{3}\,y^2)$, which are when $y = 0$ and $y = \pm\sqrt{3}$. With $f(y) = \frac{1}{2}\,y - \frac{1}{6}\,y^3$, we have $f'(y) = \frac{1}{2} - \frac{1}{3}\,y^2$, so $f'(0) > 0$ and $f'(\pm\sqrt{3}) < 0$, indicating that $y = 0$ is unstable and $y = \pm\sqrt{3}$ are asymptotically stable. This is indicated in the phase line shown below.



This suggests that every positive initial condition will eventually converge to $y = \sqrt{3}$, and every negative initial condition will converge to $y = -\sqrt{3}$. (And, of course, an initial condition of $y = 0$ will remain there.)

6. [14 points] Suppose that the phase portrait to the right is the phase portrait for a system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 constant, real-valued matrix. If the system is obtained by rewriting a second order equation as a system of first-order equations, give a possible matrix for $\mathbf{A}$. Explain how you know your choice is correct.



Solution: Because the system is a rewritten second order equation, we must have started with an equation $x'' - bx' - ax = 0$; taking $x' = y$, the resulting system is $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. We know that the eigenvalues of the coefficient matrix must be complex-valued (because the phase portrait shows a spiral) and have a positive real part (because trajectories move out from the origin). The eigenvalues of our matrix are given by $-\lambda(b-\lambda) - a = \lambda^2 - b\lambda - a = 0$ For convenience, let $b = 2$. Then the characteristic equation is $\lambda^2 - 2\lambda - a = (\lambda - 1)^2 - (a+1) = 0$, and if $a < 1$ we have complex roots. For example, if $a = -2$, we have $\lambda = 1 \pm i$. Our matrix could therefore be

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}.$$

(Note that at $(1, 0)$, this gives $\mathbf{x}' = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$, a trajectory moving straight down past the x_1 axis, as shown in the figure.)

More generally, because $\lambda^2 - b\lambda - a = 0$ we have $\lambda = \frac{1}{2}\,b \pm \frac{1}{2}\sqrt{b^2 + 4a}$, and therefore we must have $b > 0$ and $b^2 + 4a < 0$, so that $a < -\frac{1}{4}\,b^2$. Any matrix satisfying these conditions will give us the desired phase portrait.

7. [12 points] Suppose that the matrix $\mathbf{A}$ has eigenvalues $\lambda = -1$ and $\lambda = -2$, with corresponding eigenvectors $\mathbf{v}_{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_{-2} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If the solution to $\mathbf{A}\mathbf{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is $\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, sketch the phase portrait for the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Explain how you get your answer.

Solution: First note that if $\mathbf{A}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, then $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is the equilibrium solution of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \begin{pmatrix} -2 \\ 2 \end{pmatrix}$. Thus if we let $\mathbf{x} = \mathbf{x}_0 + \mathbf{u}$, then $\mathbf{u}' = \mathbf{A}\mathbf{u}$.

The phase portrait for the system $\mathbf{u}' = \mathbf{A}\mathbf{u}$ is obtained as follows. There are two negative real eigenvalues, so along lines through the origin given by the two eigevectors all trajectories move to the origin. Away from these, trajectories will collapse fastest in the direction of the vector $\mathbf{v}_{-2}$, and then approach the origin along $\mathbf{v}_{-1}$.

Then to get from $\mathbf{u}$ to $\mathbf{x}$, we add $\mathbf{x}_0$. We therefore obtain the phase portrait shown below.