Math 216 — Second Midterm

19 March, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, use Laplace transforms, **not** some other solution technique.

a. [8 points] $y'' + 2y' + y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

Solution: Applying the forward transform, letting $Y = \mathcal{L}\{y\}$, we have $(s^2+2s+1)Y-1 = \frac{1}{s+1}$, so that

$$Y = \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}.$$

Note that, using rules (2) and (B) from the table, we have $\mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} = \mathcal{L}^{-1}\{-\frac{d}{ds}\frac{1}{(s+1)}\} = te^{-t}$, and $\mathcal{L}^{-1}\{\frac{1}{(s+1)^3}\} = \mathcal{L}^{-1}\{-\frac{1}{2}\frac{d}{ds}\frac{1}{(s+1)^2}\} = \frac{1}{2}t^2e^{-t}$. Thus,

$$\begin{aligned} y &= \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\} + \mathcal{L}^{-1}\{\frac{1}{(s+1)^3}\} \\ &= te^{-t} + \frac{1}{2}t^2e^{-t}. \end{aligned}$$

b. [7 points] $y'' + 6y' + 13y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: Applying the forward transform, letting $Y = \mathcal{L}\{y\}$, we have $(s^2+6s+13)Y - s - 6 = 0$, so that

$$Y = \frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}.$$

We can invert both of these with rules (4), (5), and (C). We have

$$\begin{aligned} y = \mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4} + \frac{3}{(s+3)^2+4}\right\} \\ &= e^{-3t}\cos(2t) + \frac{3}{2}e^{-3t}\sin(2t). \end{aligned}$$

2. [14 points] Fill in the missing portions of each of the following transforms. Briefly explain how you obtain your work.

a. [7 points] $\mathcal{L}\{\begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 5 \\ e^{-(t-5)}, & t \ge 5 \end{cases}\} = \frac{1}{s}(e^{-s} - e^{-5s}) +$ $\underline{\quad e^{-5s}\frac{1}{s+1} \quad}$

Solution: Letting f be the indicated funcion and breaking the integral on the piecewise definitions, we have

$$\mathcal{L}\{f\} = \int_1^5 e^{-st}\,dt + \int_5^\infty e^{-(t-5)}e^{-st}\,dt.$$

The first of these integrals gives the terms provided. The second is

$$\int_5^\infty e^5 e^{-(s+1)t}\,dt = \lim_{b\to\infty}(-\frac{e^5}{s+1})e^{-(s+1)t}\Big|_{t=5}^{t=b} = \frac{e^5}{s+1}\,e^{-5s-5} = \frac{e^{-5s}}{s+1}.$$

b. [7 points] $\mathcal{L}^{-1}\{\frac{1}{s(s+1)(s^2+1)}\} = 1 - \frac{1}{2}\cos(t) +$ $\underline{\quad -\frac{1}{2}\sin(t) - \frac{1}{2}e^{-t} \quad}$

Solution: We can rewrite this with partial fractions:

$$\frac{1}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

The inverse transforms of these will be, in order, A, Be^{-t}, $C\cos(t)$, and $D\sin(t)$. Thus from the provided partial answer we know that $A = 1$ and $C = -\frac{1}{2}$. Then, clearing the denominators and using these values, we have

$$1 = (s+1)(s^2+1) + Bs(s^2+1) + (-\frac{1}{2}\,s + D)s(s+1).$$

If $s = -1$, we have $1 = -2B$, so that $B = -\frac{1}{2}$. If $s = 1$, we have $1 = 4-1+(-\frac{1}{2}+D)(2) = 2 + 2D$. Thus $D = -\frac{1}{2}$, and we have the indicated result.

3. [14 points] Find each of the following, as indicated.

a. [7 points] If a function $f(t)$ has the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$, use the integral definition of the Laplace transform to find the transform $\mathcal{L}\{\int_0^t f(x)\,dx\}$ in terms of $F(s)$. *(You may assume that $\int_0^\infty f(x)\,dx = L$, a finite value.)*

Solution: We have, integrating by parts with $u = \int_0^t f(x)\,dx$ (so that $u' = f(t)$) and $v' = e^{-st}$ (so that $v = -\frac{1}{s}e^{-st}$),

$$\begin{aligned}
\mathcal{L}\{\int_0^T f(t)\,dt\} &= \int_0^\infty (\int_0^t f(x)\,dx)\,e^{-st}\,dt \\
&= -\frac{1}{s}\lim_{b\to\infty}(\int_0^t f(x)\,dx)\,e^{-st}\Big|_{t=0}^{t=b} + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt \\
&= -\frac{1}{s}\int_0^0 f(x)\,dx + \lim_{b\to\infty}\frac{1}{s}L\,e^{-sb} + \frac{1}{s}F(s).
\end{aligned}$$

The first term on the right-hand side is an integral over an interval of zero length, and so is zero, and in the limit as $b \to \infty$, the second vanishes because of the negative exponential e^{-sb}. Thus we have $\mathcal{L}\{\int_0^t f(x)\,dx\} = \frac{1}{s}\,F(s)$.

b. [7 points] Find an explicit expression for $Y = \mathcal{L}\{y\}$ if $y''' + 3y = t^2e^{-4t} - e^{-2t}\cos(5t)$. *(Note that you are not asked to solve the differential equation.)*

Solution: Because of the linearity of the transform, we can calculate the transform of each term separately. We have $\mathcal{L}\{y'''\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$ and $\mathcal{L}\{3y\} = 3Y$. To find $\mathcal{L}\{t^2e^{-4t}\}$, we note that $\mathcal{L}\{e^{-4t}\} = \frac{1}{s+4}$, so that, by rule (B), $\mathcal{L}\{t^2e^{-4t}\} = \frac{d^2}{ds^2}\frac{1}{s+4} = \frac{2}{(s+4)^3}$. To find $\mathcal{L}\{e^{-2t}\cos(5t)\}$, we use rule (C) and the transform $\mathcal{L}\{\cos(5t)\} = \frac{s}{s^2+25}$ to get $\mathcal{L}\{e^{-2t}\cos(5t)\} = \frac{s+2}{(s+2)^2+25}$. Putting these all together and solving for Y, we have

$$Y = \frac{s^2y(0) + sy'(0) + y''(0)}{s^3+3} + \frac{2}{(s+4)^3(s^3+3)} - \frac{s+2}{((s+2)^2+25)(s^3+3)}.$$

4. [15 points] Find explicit, real-valued solutions for each of the following, as indicated. Do **not** use Laplace transform techniques on this problem.

a. [8 points] Find the general solution to $y'' + 2y' + 4y = e^{-t} + t^2$.

Solution: We first look for the general solution in the form $y = y_c + y_p$. For y_c, the guess $y = e^{rt}$ gives $r^2 + 2r + 4 = (r+1)^2 + 3 = 0$, so that $r = -1 \pm \sqrt{3}\,i$, and $y_c = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$.

To find y_p, we use the method of undetermined coefficients, looking for each of the forms on the right-hand side separately. For the e^{-t} term, we guess $y = ae^{-t}$, so that $(1 - 2 + 4)a = 1$, and $a = \frac{1}{3}$. For the t^2 term, we guess $y = a_0 + a_1 t + a_2 t^2$, so that $2a_2 + 2(2a_2 t + a_1) + 4(a_2 t^2 + a_1 t + a_0) = t^2$. Collecting powers of t, we have $a_2 = \frac{1}{4}$, $a_1 = -\frac{1}{4}$, and $a_0 = 0$.

The general solution to the problem is therefore

$$y = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t) + \frac{1}{3}e^{-t} + \frac{1}{4}t^2 - \frac{1}{4}t.$$

b. [7 points] Solve $y'' + 5y' + 4y = 3\cos(2t)$, $y(0) = 0$, $y'(0) = 1$

Solution: We look for the general solution as $y = y_c + y_p$. For y_c we have $y = e^{rt}$, and $r^2 + 5r + 4 = (r+4)(r+1) = 0$, so that $y_c = c_1 e^{-4t} + c_2 e^{-t}$.

For y_p we use the method of undetermined coefficients guess $y_p = a\cos(2t) + b\sin(2t)$. Plugging in, we have

$$-4a\cos(2t) - 4b\sin(2t) - 10a\sin(2t) + 10b\cos(2t) + 4a\cos(2t) + 4b\sin(2t) = 3\cos(2t).$$

Collecting terms in $\cos(2t)$ and $\sin(2t)$, we have $10b = 3$, and $-10a = 0$; thus $a = 0$ and $b = \frac{3}{10}$. Our general solution is therefore

$$y = c_1 e^{-4t} + c_2 e^{-t} + \frac{3}{10}\sin(2t).$$

Applying the initial conditions, we have $y(0) = c_1 + c_2 = 0$, and $y'(0) = -4c_1 - c_2 + \frac{3}{5} = 1$. Substituting the first into the second, we have $3c_2 = \frac{2}{5}$, so that $c_2 = \frac{2}{15}$ and $c_1 = -\frac{2}{15}$. Thus

$$y = -\frac{2}{15}\,e^{-4t} + \frac{2}{15}\,e^{-t} + \frac{3}{10}\sin(2t).$$

5. [14 points] Consider the operators $T[y] = yy'' + 2y^2y'$ and $U[y] = t^2y'' - ty' - 3y$.

a. [9 points] Show that T is nonlinear while U is linear.

Solution: We note that

$$T[cy] = cy(cy'') - 2c^2y^2(cy') = c^2(yy'' - 2cy^2y') \neq cT[y]$$

(because $cT[y] = c(yy'' + 2y^2y')$). Therefore T is not linear. (Other possible arguments include the calculations $T[y_1 + y_2] \neq T[y_1 + y_2]$, and $T[c_1y_1 + c_2y_2] \neq c_1T[y_1] + c_2T[y_2]$.) However,

$$\begin{aligned} U[c_1y_1 + c_2y_2] &= t^2(c_1y_1'' + c_2y_2'') - t(c_1y_1' + c_2y_2') - 3(c_1y_1 + c_2y_2) \\ &= c_1(t^2y_1'' - ty_1' - 3y_1) + c_2(t^2y_2'' - ty_s' - 3y_2) \\ &= c_1U[y_1] + c_2U[y_2]. \end{aligned}$$

Thus U is linear.
Alternately, we said that a linear operator may also be written in the form $L = D^2 + p(t)D + q(t)$, so that $L[y] = D^2[y] + p(t)D[y] + q(t)y$. For T, we have

$$T[y] = yy'' + 2y^2y' = yD^2[y] + 2y^2D[y] = y(D^2[y] + 2yD[y]),$$

which is not in the form of a linear operator, while

$$U[y] = t^2(D^2[y] - t^{-1}D[y] - 3y) = t^2(D^2 - t^{-1}D - 3t^{-2})[y],$$

so that the operator $U = t^2D^2 - tD - 3$. Note that this isn't in the form of L; it is an easy generalization from that, but isn't quite consistent with our understanding. It's best to make the argument above, or to consider the operator in the context of a differential equation: the equation $U[y] = 0$ can be written as $L[y] = 0$, with $L = D^2 - t^{-1}D - 3t^{-2}$, a linear operator.

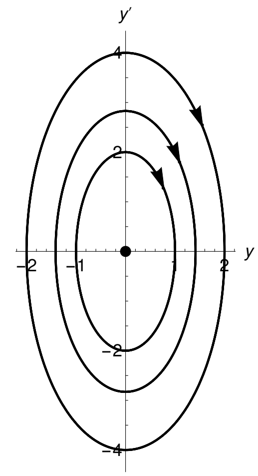
b. [5 points] Show that $y_1 = t^{-1}$ and $y_2 = t^3$ constitute a fundamental set of solutions to the equation $U[y] = 0$. What is the general solution to $U[y] = 0$?
(You may assume that $t > 0$.)

Solution: A fundamental solution set to a linear second-order homogeneous equation consists of two linearly independent solutions. Note that $U[y_1] = t^2(2t^{-3}) - t(-t^{-2}) - 3t^{-1} = 3t^{-1} - 3t^{-1} = 0$, and $U[y_2] = t^2(6t) - t(3t^2) - 3t^3 = 6t^3 - 6t^3 = 0$, so y_1 and y_2 are solutions to $U[y] = 0$. Then

$$W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^3 \\ -t^{-2} & 3t^2 \end{vmatrix} = 3t + t = 4t \neq 0,$$

so the solutions are linearly independent. Thus the general solution is $y = c_1t^{-1} + c_2t^3$.

6. [13 points] Consider the phase portrait shown to the right, which shows the phase portrait for a linear, second-order, constant coefficient, homogeneous differential equation $L[y] = 0$.



a. [7 points] Write a differential equation that could give this phase portrait. Explain how you obtain your solution, and why is it correct.

Solution: We note that the phase portrait shows a center, that is, trajectories are simple closed loops. This suggests that the solutions are sines and cosines, so that we should have an equation $y'' + \omega_0^2 y = 0$. Solutions to this are $y_1 = \cos(\omega_0 t)$ and $y_2 = \sin(\omega_0 t)$, so that the phase plane trajectories are given by $\mathbf{x}_1 = c_1 \begin{pmatrix} \cos(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) \end{pmatrix}$ and $\mathbf{x}_2 = c_2 \begin{pmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{pmatrix}$. We note that the vertical stretch of the shown trajectories appears to be twice that of the horizontal, so guess that $\omega_0 = 2$. Our equation is therefore $y'' + 4y = 0$.

b. [6 points] Suppose that we add a forcing term $f(t) = \cos(15t/8)$ to the equation, so that we are solving $L[y] = f(t)$. Sketch an approximate solution curve with $y(0) = 0$, $y'(0) = 1$. Explain why your solution appears as it does.

Solution: Note that the forcing frequency $\omega = 15/8$ is close to the natural frequency of the system, $\omega_0 = 2$. So we expect to see at least a mild beats phenomenon. This is shown in the graph below.

7. [15 points] In our lab on lasers, we considered a linearization of the nonlinear model for the population inversion N and light intensity P. A critical point of the nonlinear system is $(N, P) = (1, A-1)$, and linearizing the system near this gives the linear system

$$u' = -\gamma(Au + v), \qquad v' = (A-1)u,$$

where γ and A are constants.

a. [5 points] Rewrite this as a single, second-order equation in v.

Solution: Note that, from the second equation, $u = \frac{1}{A-1}\,v'$. Plugging this into the first equation, we have

$$\frac{1}{A-1}\,v'' + \frac{\gamma A}{A-1}\,v' + \gamma v = 0,$$

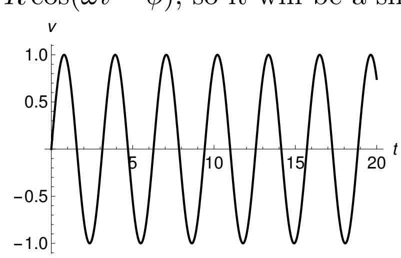
or $v'' + \gamma A\,v' + \gamma(A-1)v = 0$.

b. [5 points] Suppose that for some α and β your equation from (a) is $v'' + \alpha v' + \beta v = 0$. Under what conditions on α and β will the solution for v be underdamped? Write down two real-valued linearly independent solutions to the equation in this case.

Solution: We note that solutions to this equation are $v = e^{rt}$ with $r = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$. This will give underdamping if $\alpha^2 - 4\beta < 0$, that is, if $\alpha^2 < 4\beta$. In terms of the constants we obtained in (a), this is $\gamma^2 A^2 < 4\gamma(A-1)$, or $\gamma < 4\frac{A-1}{A}^2$. The two solutions are $y_1 = e^{-\mu t}\cos(\nu t)$ and $y_2 = e^{-\mu t}\sin(\nu t)$, where $\mu = \frac{\alpha}{2} = \frac{\gamma A}{2}$ and $\nu = \frac{1}{2}\sqrt{4\beta - \alpha^2} = \frac{1}{2}\sqrt{4\gamma(A-1) - \gamma^2 A^2}$.

c. [5 points] Now suppose that we force the underdamped equation given in (b) with the periodic forcing term $f(t) = \cos(\omega t)$. Sketch a graph of the steady state solution of the problem. Explain why your graph has the form it does. If ω changes from very small to very large values, how would you expect your sketch to change? Explain.

Solution: The steady state solution will be the response to the forcing, because (as we see in (b)) the non-forced response decays to zero. Because y_1 and y_2 do not have the same form as $f(t)$, we know that the steady state (particular) solution will have the form $v_p = a\cos(\omega t) + b\sin(\omega t) = R\cos(\omega t - \phi)$, so it will be a simple sinusoid:



If we vary ω, we expect that the frequency of the solution will change, and that its amplitude will also change. A reasonable guess is that the amplitude will initially increase, obtain a local maximum at an ω near $\nu = \frac{1}{2}\sqrt{4\beta - \alpha^2} = \frac{1}{2}\sqrt{4\gamma(A-1) - \gamma^2 A^2}$, and then decay to zero as ω becomes very large.