Math 216 — Final Exam

19 April, 2018

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions for each of the following, as indicated.

a. [6 points] Solve for y if $y' + ye^t - e^t = 0$, $y(0) = 0$.

Solution: This is both linear and separable. If we proceed with an integrating factor, we write $y' + (e^t)y = e^t$, so that $\mu = e^{\int e^t\,dt} = e^{e^t}$. Then $(ye^{e^t})' = e^te^{e^t}$, and, integrating both sides, we have $ye^{e^t} = e^{e^t} + C$. Solving for y, $y = 1 + Ce^{-e^t}$. The initial condition requires $0 = 1 + Ce^{-1}$, so $C = -e$, and

$$y = 1 - e^{1-e^t}.$$

Alternately, we can separate variables. To do this, we rewrite the equation as $y' = -e^t(y-1)$, so that $y'/(y-1) = -e^t$. Integrating both sides we have $\ln|y-1| = -e^t + C'$. Exponentiating both sides and letting $C = \pm e^{C'}$, we have $y - 1 = Ce^{-e^t}$, so that $y = 1 - Ce^{-e^t}$, as before. We can then apply the initial condition as before.
We could, of course, also apply the initial condition to the relation $\ln|y-1| = -e^t + C'$. Then $0 = -1 + C'$ so that $C' = 1$, and the final solution is as we expect, $y = 1 - e^{1-e^t}$.

b. [9 points] Find the general solution to $\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}\mathbf{x}$.

Solution: We know that solutions will be of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$, where λ and $\mathbf{v}$ are the eigenvalues and eigenvectors of the coefficient matrix. Here, eigenvalues satisfy $(3-\lambda)^2 + 4 = 0$, so $\lambda = 3 \pm 2i$. With these values, components of the eigenvector satisfy $\mp 2iv_1 + 2v_2 = 0$, so $v_2 = \pm iv_1$, and eigenvectors are $\mathbf{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. To find a real-valued solution we use the real and imaginary parts of the exponential solution as our fundamental solution set, so that

$$\mathbf{x} = c_1 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} e^{3t}.$$

2. [15 points] Find explicit, real-valued solutions for each of the following, as indicated.

a. [7 points] Find the general solution to $y''' + 4y'' + 3y' = 5 - e^{-2t}$.

Solution: For the complementary homogeneous solution, we look for $y = e^{\lambda t}$, so that $\lambda(\lambda^2 + 4\lambda + 3) = \lambda(\lambda + 1)(\lambda + 3) = 0$. Thus we have $\lambda = 0$, $\lambda = -1$, and $\lambda = -3$, so that the complementary homogeneous solution is $y = c_1 + c_2 e^{-t} + c_3 e^{-3t}$. For the particular solution, noting that a constant is part of the homoegenous solutions, we use the method of undetermined coefficients and guess $y_p = y_{p1} + y_{p2} = at + be^{-2t}$. For the first part of this, we have $3a = 5$, so $a = 5/3$. For the second, we have $(-8 + 16 - 6)b = -1$, and $b = -1/2$. The general solution is therefore

$$y = c_1 + c_2 e^{-t} + c_3 e^{-3t} + \frac{5}{3}t - \frac{1}{2}\,e^{-2t}.$$

b. [8 points] Solve $y'' + 3y' + 2y = 4u_1(t) - 3\delta(t - 2)$, $y(0) = 0$, $y'(0) = 1$.

Solution: While it is possible to solve this with other methods, by far the best approach is to use Laplace transforms. Taking the forward transform of both sides and letting $Y = \mathcal{L}\{y\}$, we have $s^2 Y - 1 + 3sY + 2Y = \frac{4}{s}\,e^{-s} - 3e^{-2s}$, so that

$$Y = \frac{1}{(s+1)(s+2)} + \frac{4}{s(s+1)(s+2)}e^{-s} - \frac{3}{(s+1)(s+2)}e^{-2s}.$$

To invert the first and last term in Y, we rewrite $\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$, so that $1 = A(s+2) + B(s+1)$. Taking $s = -1$ and $s = -2$, we have $A = 1$, $B = -1$. Similarly, for the middle term we write $\frac{1}{s(s+1)(s+2)} = \frac{C}{s} + \frac{D}{s+1} + \frac{E}{s+2}$. Then $1 = C(s+1)(s+2) + Ds(s+2) + Es(s+1)$. With $s = 0$, -1, and -2, we have $C = \frac{1}{2}$, $D = -1$, and $E = \frac{1}{2}$. Thus

$$\begin{aligned} y = \mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)(1 - 3e^{-2s}) + \left(\frac{1}{2s} + \frac{1}{s+1} + \frac{1}{2(s+2)}\right)4e^{-s}\} \\ &= (e^{-t} - e^{-2t}) - 3(e^{-(t-2)} - e^{-2(t-2)})u_2(t) \\ &\qquad + (2 - 4e^{-(t-1)} + 2e^{-2(t-1)})u_1(t). \end{aligned}$$

3. [15 points] Consider the differential equation $V' = V^{1/3}(k - V^{2/3})$, where $V(t)$ is some (real-valued) physical quantity and k is a constant.

a. [5 points] Find all equilibrium solutions of the equation and their stability. How does the number of equilibrium solutions depend on k?

Solution: Equilibrium solutions are constant solutions, so $V' = 0 = V^{1/3}(k - V^{2/3})$. Thus $V = 0$ is an equilibrium solution for any value of k, and $V = \pm k^{3/2}$ is an equilibrium solution if $k > 0$. So there is one equilibrium solution if $k \leq 0$, and three if $k > 0$.

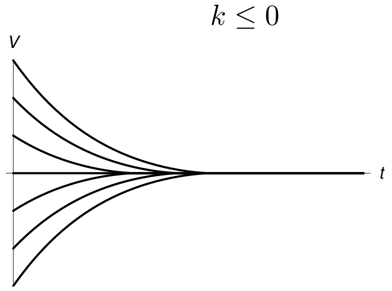
To analyze stability it is easiest to consider the sign of the right-hand side of the equation, $f(V) = V^{1/3}(k - V^{2/3})$. If $k \leq 0$, this is a negative number times $V^{1/3}$, so that $f(V)$ is positive for $V < 0$ and negative for $V > 0$, so that $V = 0$ is an asymptotically stable equilibrium.

If $k > 0$ there are three critical points. If $V > (k^{3/2})$, $f(V) < 0$, and $f(V)$ changes sign at each critical point. Thus $V = \pm k^{3/2}$ are asymptotically stable, and $V = 0$ is unstable.

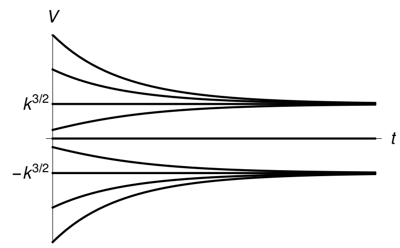
b. [5 points] Sketch representative solution curves for the equation. Note that you may need more than one graph if you found in (a) a different number of equilibrium solutions depending on the values of k. In the long run, what solution(s) to the equation do you expect to see?

Solution: We have two cases, $k \leq 0$ and $k > 0$. Based on the stability analysis above, we expect solution curves to look like the graphs below. As $t \to \infty$, for $k < 0$ we expect to see only the zero solution; for $k > 0$, we expect to see one of the solutions $V = \pm k^{3/2}$.

$k \leq 0$



$k > 0$



Problem 3, continued. We are considering the differential equation $V' = V^{1/3}(k - V^{2/3})$, where $V(t)$ is some (real-valued) physical quantity and k is a constant.

c. [5 points] Are there any initial conditions $V(t_0) = V_0$ for which you might expect this differential equation could have no solution? More than one solution? Explain. *(Hint: you shouldn't need to solve the equation to answer this question.)*

Solution: Note that, with the usual assumptions about the cube root, $f(V)$ is continuous for all values of V, and $f'(V) = \frac{1}{3}V^{-2/3}(k - V^{2/3}) - \frac{2}{3}$ is continuous everywhere but at $V = 0$. Thus for all t_0 and all $V_0 \neq 0$ we are guaranteed a unique solution to the initial value problem. When $V_0 = 0$, we know there is one solution ($V = 0$), but it is not clear if there could be more than one. Thus we know there is no initial condition for which there are no solutions, and for $V_0 = 0$ we could have multiple solutions if the solution trajectories we graphs to the left, above, reach the t axis in finite time.

4. [15 points] If the solution to the initial value problem $y'' + 4y' + ay = 3\delta(t - \pi)$, $y(0) = 0$, $y'(0) = k$, is for $t < \pi$ a decaying sinusoid with a local maximum at $t = \pi/2$, and is zero for all values of $t \geq \pi$, what are k and a?
Use Laplace transforms in your solution to this problem.

Solution: Because the forcing term is a delta function we are inclined to use Laplace transforms even in the absence of the strongly worded admonition. The forward transform is, with $Y = \mathcal{L}\{y\}$, $(s^2+4s+a)Y = k+3e^{-\pi s}$, so that $Y = k/(s^2+4s+a)+3e^{-\pi s}/(s^2+4s+a)$. Next, we know that solutions are decaying sinusoids for $t < \pi$, so the denominator of this, s^2+4s+a, must not factor over reals. Therefore we can write $s^2+4s+a = (s+2)^2+(a-4)$, with $a-4 > 0$, and see that the inverse transform of the first portion of Y is $y = \frac{k}{\sqrt{a-4}}e^{-2t}\sin(\sqrt{a-4}\,t)$. We know that $y(\pi)$ must be zero, and $y(\pi/2)$ is a local maximum; the easiest way for this to be the case is for the pseudo-period to be 2π, so that $\sqrt{a-4} = 1$, which suggests $a = 5$. Thus we have

$$\begin{aligned} y &= ke^{-2t}\sin(t) + \mathcal{L}^{-1}\{\frac{3e^{-\pi s}}{(s+2)^2+1}\} \\ &= ke^{-2t}\sin(t) + 3e^{-2(t-\pi)}\sin(t-\pi)u_\pi(t). \end{aligned}$$

Next, we need this to be zero for all $t \geq \pi$. This will be the case when the magnitudes of the two terms sum to zero for $t \geq \pi$, or when $ke^{-2\pi} = 3$; thus $k = 3e^{2\pi}$.

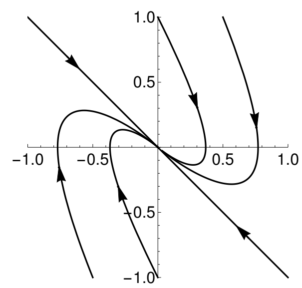
5. [12 points] Each of the following requires a short (one equation or formula) answer. Provide the required answer, and a short (one or two sentence) explanation.

a. [3 points] Write a linear, constant coefficient, second order, nonhomogeneous differential equation for which the method of undetermined coefficients is not applicable.

Solution: We must have a forcing term that is not of polynomial, exponential, or sinusoidal function, or a product of those. One such example is

$$y'' + y = \tan(t).$$

b. [3 points] Write a linear, constant coefficient, second order differential equation that has the phase portrait shown to the right.



Solution: The phase portrait shows a repeated eigenvalue (root of the characteristic equation), so this should be something like

$$y'' + 2y' + y = 0.$$

This has the solutions $y_1 = e^{-t}$ and $y_2 = te^{-t}$, so the eigenvector is $\begin{pmatrix} 1 & -1 \end{pmatrix}^T$ $(= \begin{pmatrix} y_1(0) & y_1'(0) \end{pmatrix}^T)$, as shown. (Note also that the second solution is $\mathbf{x} = (t\mathbf{v} + \begin{pmatrix} 0 & 1 \end{pmatrix}^T)e^{-t}$, which completes the phase portrait shown.)

c. [3 points] If $L[y] = f(t)$ is a linear, constant coefficient, second order differential equation and $L[y] = 0$ is solved by $y = c_1e^{-t} + c_2te^{-t}$, write a function $f(t)$ for which a good solution guess would be $y = At^3e^{-t} + Bt^2e^{-t}$.

Solution: Note that if $f(t) = e^{-t}$, we must guess $y = At^2e^{-t}$ to avoid having any term in the guess that is part of the homogeneous solution. Therefore we can take

$$f(t) = kte^{-t},$$

for any k; our guess is then $y = t^2(Ate^{-t} + Be^{-t})$, as desired.

d. [3 points] Write a linear, constant coefficient, second order differential equation having a phase portrait that is a spiral sink converging on the point $(2, 0)$.

Solution: For the phase portrait to show a spiral sink the characteristic equation must have complex conjugate roots with negative real part. One such is $r^2 + 2r + 2 = 0$ (for which $r = -1 \pm i$), so that the linear operator is $L = D^2 + 2D + 2$. Then, for the critical point to be $(2, 0)$, we must have an equilibrium solution $x = 2$. The equation could therefore be

$$x'' + 2x' + 2x = 4.$$

6. [12 points] Consider the system given by $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} -3 & 2 & 3 \\ 0 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$. Eigenvalues of $\mathbf{A}$ are $\lambda = -4$, -3, and 2, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, respectively.

a. [4 points] Give an initial condition for which trajectories converge to the origin. Explain how you know your answer is correct.

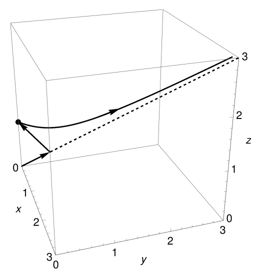
Solution: The general solution to the problem is $\mathbf{x} = c_1\mathbf{v}_1e^{-4t} + c_2\mathbf{v}_2e^{-3t} + c_3\mathbf{v}_3e^{2t}$. Thus any initial condition that is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ will converge to the origin. One such is $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -\mathbf{v}_1$.

b. [4 points] Give all initial conditions for which the resulting trajectories remain bounded for all t. Explain.

Solution: Given the general solution in (a), any initial condition that includes the third eigenvector will diverge to infinity. Thus, the initial conditions that remain bounded are $\mathbf{x}(0) = \mathbf{x}_0 = a\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, for any (real-valued) a and b.

c. [4 points] Suppose that $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Describe the solution trajectory in phase space. What does it look like as $t \to \infty$? Explain.

Solution: We note that this initial condition is one half the sum of $\mathbf{v}_1$ and $\mathbf{v}_3$. Thus the solution in this case is $\mathbf{x} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}e^{-4t} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t} = \frac{1}{2}\mathbf{v}_1e^{-4t} + \frac{1}{2}\mathbf{v}_3e^{2t}$. As t increases, the first will decay to zero (very fast), and we will be left with an outward trajectory along the line $\mathbf{x} = s\mathbf{v}_3$. Thus we will have a trajectory in the plane determined by the two eigenvectors, that will have a hyperbolic appearance: tracing forward it asymptotes to the line indicated; tracing backwards it would asymptote to the line through the origin along the other eigenvector. This is shown in the figure below, with the components of the intial condition contributed by each eigenvector shown, and the initial condition indicated by a point.



7. [16 points] Our model for a ruby laser is, with N = the population inversion of atoms and P = the intensity of the laser,

$$N' = \gamma A - \gamma N(1+P), \quad P' = P(N-1).$$

In lab we found that the critical points of this system are $(N,P) = (A,0)$ and $(N,P) = (1, A-1)$. For this problem we will assume that $\gamma = \frac{1}{2}$; A is, of course, also a constant.

a. [4 points] Find a linear system that approximates the nonlinear system near the critical point $(A,0)$. Show that if $A < 1$ this critical point is asymptotically stable, and if $A > 1$ it is unstable.

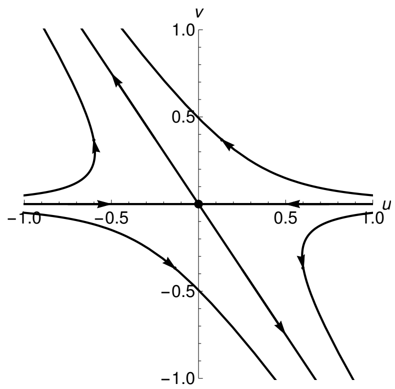
Solution: The easiest way to find the linearization is to use the Jacobian. Here we have $N' = F(N,P) = \frac{A}{2} - \frac{1}{2}(N + NP)$, $P' = G(N,P) = PN - P$, so that

$$J = \begin{pmatrix} F_N & F_P \\ G_N & G_P \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1+P) & -\frac{1}{2}N \\ P & N-1 \end{pmatrix}.$$

At $(A,0)$, this is $J(A,0) = \begin{pmatrix} -\frac{1}{2} & -\frac{A}{2} \\ 0 & A-1 \end{pmatrix}$, which has eigenvalues $\lambda = -\frac{1}{2}$ and $\lambda = A-1$. Thus, if $A < 1$ both eigenvalues are real and negative, and the critical point is asymptotically stable; if $A > 1$, the second eigenvalue is positive and the critical point becomes unstable.

b. [6 points] Suppose that the linear system you obtained in (a) is, for some value of A, $u' = -\frac{1}{2}u - v$, $v' = v$. Sketch a phase portrait that shows solution trajectories of the linear system. Explain how these trajectories are related to trajectories in the (N,P) phase plane.

Solution: Note that this is $\mathbf{u}' = \begin{pmatrix} -\frac{1}{2} & -1 \\ 0 & 1 \end{pmatrix}\mathbf{u}$, so this is apparently the result we obtained in (a) with $A = 2$. Eigenvalues of the coefficient matrix are $\lambda = -\frac{1}{2}$ and $\lambda = 1$. When $\lambda = -\frac{1}{2}$ we have the eigenvector $\mathbf{v} = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, and when $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} -2 & 3 \end{pmatrix}^T$. These give the saddle point shown below.



These trajectories will be very similar to the trajectories in the (N,P) plane at the critical point, $(A,0)$.

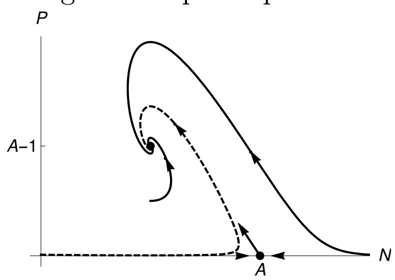
Problem 7, cont. We are considering the system

$$N' = \frac{1}{2}A - \frac{1}{2}N(1+P), \quad P' = P(N-1),$$

which has critical points $(N,P) = (A,0)$ *and* $(N,P) = (1, A-1)$.

c. [6 points] Suppose that, for the value of A used in (b), the coefficient matrix for the linear system approximating (N,P) near the critical point $(1, A-1)$ is $\begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$, which has eigenvalues $\lambda = \frac{1}{2}(-1 \pm i)$. Using this information with your work in (b), sketch a representative solution curve for P as a function of t, if $P(0) = 0.01$ when $N(0) = 0$.

Solution: We note that in the phase plane for the nonlinear system, critical points are at $(A,0)$ and $(1, A-1)$. The phase portrait near the former is given in (b); for the latter, we know it is a spiral sink, and, because $\begin{pmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, the inward spiral must be counter clockwise. This gives the phase portrait shown below.



A trajectory starting at $(0, 0.01)$ is suggested by the dashed curve. Reading the behavior of P from this, we get the curve below. We know that it starts at $(0, 0.01)$, that it must remain close to the t-axis for a while, and then must oscillate around and converge to the line $P = A - 1$. Finally, note that we do not know the time scale on which these transitions take.