Math 216 — First Midterm

20 February, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

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1. [15 points] Solve each of the following, finding explicit real-valued solutions as indicated.

a. [7 points] Find the general solution to $y' = \dfrac{5 + 5s^5 - 5s^4 y}{1 + s^5}$.

b. [8 points] Solve the initial value problem $R' = (2 - 10z)R^2$, $R(0) = -2$.

2. [15 points] Solve each, finding explicit real-valued solutions as indicated.

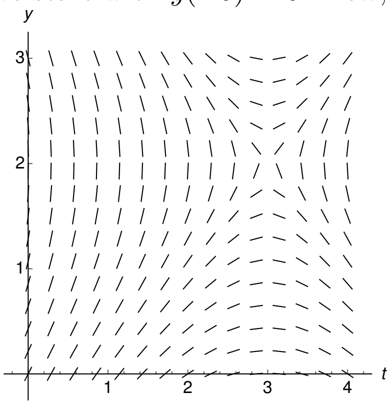
a. [8 points] Solve the initial value problem $x' = -y$, $y' = 12x - 7y$, $x(0) = 2$, $y(0) = 1$.

b. [7 points] Find the general solution to $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 6 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

3. [12 points] Suppose we are solving the initial value problem $y' = \dfrac{t-3}{y-2}$, $y(0) = y_0$.

a. [6 points] A direction field for the differential equation is shown to the right, below. Using this and your knowledge of the differential equation, explain what the solution will look like if we start with the initial condition $y(0) = 0$, and if we start with $y(1.5) = 0$. How, and why, are these solutions different?

(The printed exam copy had $y(1) = 0$ for the second initial condition. This was supposed to be $y(1.5)$; through $(0, 1)$ the solution is $y = t - 1$.)



b. [6 points] Solve the problem with initial condition $y(0) = 0$. Based on your solution, for what values of t and y does your solution exist? How is this related to the existence and uniqueness theorem?

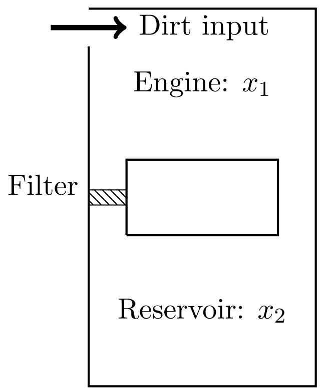
4. [15 points] Consider the system of differential equations given by $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

a. [4 points] If $\mathbf{P}(t) = \begin{pmatrix} 0 & 1 \\ -2t^{-2} & 2t^{-1} \end{pmatrix}$, is this a linear or nonlinear problem? If we apply the initial condition, will there be a unique solution? Explain.

b. [6 points] If $\mathbf{P}(t) = \mathbf{A}$, a 2×2 constant real-valued matrix, and if a general solution to the system is $\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2(t\mathbf{v}_1 + \mathbf{v}_2)e^{\lambda t}$, how many solutions are there to each of the following algebraic systems of equations? Why?
(i) $\mathbf{A}\mathbf{x} = 2\lambda\mathbf{x}$
(ii) $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1$

c. [5 points] If $\mathbf{P}(t) = \mathbf{B}$, a 2×2 constant real-valued matrix, and a solution to the system is $\mathbf{x} = \begin{pmatrix} \cos(3t) \\ \cos(3t) - 2\sin(3t) \end{pmatrix} e^{-4t}$, what are the eigenvalues and eigenvectors of $\mathbf{B}$?

5. [16 points] In internal combustion engines, oil is circulated from a reservoir, around moving parts to lubricate them, and back to the reservoir. As it circulates, it collects dirt from the engine. To remove the dirt, oil from the reservoir is passed through a filter. A simple model for this system is shown to the right. Dirt is "added" to the oil in the engine, and we denote the amount of dirt in the engine compartment as x_1 and that in the reservoir as x_2. Suppose that the amount of oil in the engine compartment is 3 quarts and in the reservoir there are 2 quarts. Oil moves from the engine to the reservoir and back at a rate of 1 quart/minute, and the filter removes a fraction of the dirt from the oil returning from the reservoir to the engine.



a. [4 points] Suppose that x_1 and x_2 are measured in grams. A model for the amount of dirt in either compartment is

$$x_1' = -\frac{1}{3}\,x_1 + \frac{3}{5}\left(\frac{1}{2}\right)x_2 + 3, \quad x_2' = \frac{1}{3}\,x_1 - \frac{1}{2}\,x_2.$$

How much dirt is added to the oil in the engine? Why is there the term $\frac{1}{3}\,x_1$ in each equation, and why does it have this form? How much of the dirt in the oil is removed by the filter, and how do you know?

b. [4 points] Find the equilibrium solution(s) for this system. What is the physical meaning of the equilibrium solution?

Problem 5, continued. We are considering the system

$$x_1' = -\frac{1}{3}\,x_1 + \frac{3}{5}\left(\frac{1}{2}\right)x_2 + 3, \quad x_2' = \frac{1}{3}\,x_1 - \frac{1}{2}\,x_2.$$

c. [4 points] The eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -1/3 & 3/10 \\ 1/3 & -1/2 \end{pmatrix}$ are, approximately, $\lambda_1 = -0.75$ and $\lambda_2 = -0.1$, with $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Sketch a phase portrait for this system.

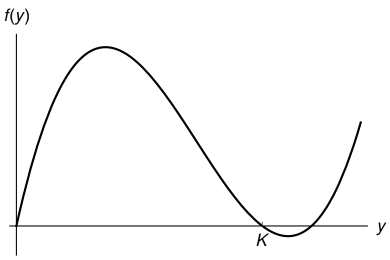
d. [4 points] Suppose that, somehow, we start with the initial condition $x_1(0) = 22.5$ and $x_2(0) = 0$. Use your work in (b) to sketch, approximately, what you expect x_1 and x_2 to look like as functions of time.

6. [15 points] In lab 1 we considered the Gompertz equation, $y' = r\,y\,\ln(K/y)$. We explore this further in this problem.

a. [5 points] Consider the initial condition $y(0) = 1$. Find a linear approximation to the Gompertz equation that is valid near this initial condition. Under what conditions would you expect your approximation to be accurate?

b. [5 points] We found that for y near K, the Gompertz equation is approximated as $y' = -rK(y - K)$. Solve this and explain what its solution tells us about solutions to the Gompertz equation.

c. [5 points] If we retain two terms from the Taylor expansion of $\ln(K/y)$ near $y = K$, we obtain the cubic differential equation $y' = f(y)$, where $f(y)$ is shown in the figure to the right. Sketch a phase line for this equation and explain what it suggests about the long-term behavior of the tumor.



7. [12 points] For each of the following, give an example as indicated, and a short (one or two sentence) explanation for how your example satisfies the indicated criteria.

a. [4 points] Give an example of an autonomous first-order differential equation with two equilibrium solutions, neither of which are stable.

b. [4 points] Give an example of a linear, constant-coefficient system of two differential equations whose phase portrait is a stable counterclockwise spiral.

c. [4 points] Give an example of a linear, constant-coefficient system of two differential equations that has a critical point that is not at the origin.