Math 216 — Second Midterm

27 March, 2019

This sample exam is provided to serve as one component of your studying for this exam in this course. **Please note that it is not guaranteed to cover the material that will appear on your exam, nor to be of the same length or difficulty**. In particular, the sections in the text that were covered on this exam may be slightly different from those covered by your exam.

1. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO NOT use Laplace transforms**.

a. [7 points] Find the general solution to $2U''(t) + 12U'(t) + 16U(t) = 12e^{4t}$.

b. [8 points] Find the solution to the initial value problem $y''(t) + 6y'(t) + 9y(t) = 3e^{-3t}$, $y(0) = 0$, $y'(0) = 1$.

2. [15 points] Find explicit, real-valued solutions to each of the following, as indicated. For this problem, **DO use Laplace transforms**.

a. [7 points] Find the solution to the initial value problem $y'' + 4y' + 20y = 0$, $y(0) = 1$, $y'(0) = 5$.

b. [8 points] Find the solution to the initial value problem $y'' + 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$.

3. [14 points] Suppose that $L[y] = y'' + p(t)y' + q(t)y$. *(Note that $L[y]$ here is a differential operator, not the Laplace transform $\mathcal{L}\{y\}$.)*

a. [7 points] If $L[t^2] = 2 + 2tp(t) + t^2 q(t) = 0$ and $L[t^2 \ln(t)] = (2\ln(t) + 3) + (2t\ln(t) + t)p(t) + t^2 \ln(t) q(t) = 0$, which, if any, of the following functions y are solutions to $L[y] = 0$ on the domain $t > 0$? Which, if any, give a general solution on this domain? Why? (In these expressions, c_1 and c_2 are real constants.)

$y_1 = 5t^2$	$y_2 = 5t^2(1 + 2\ln(t))$	$y_3 = c_1 t^2 + c_2 t^2 \ln(t)$
$y_4 = -t^2 \ln(t)$	$y_5 = t^4 \ln(t)$	$y_6 = c_1 t^2 (1 + \ln(t))$
$y_7 = t^{-2} \ln(t)$	$y_8 = W[t^2, t^2 \ln(t)] = t^3$	$y_9 = c_1(5t^2 - 2c_2 \ln(t))$

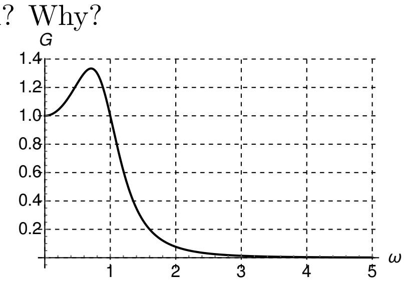
b. [7 points] Now suppose that $p(t) = 2$ and $q(t) = 10$, and let $L[y] = y'' + 2y' + 10y = g(t)$. For what $g(t)$ will the steady state solution to this problem be constant? Solve your equation with this $g(t)$ and explain how your solution confirms that your $g(t)$ is correct.

4. [15 points] In lab 2 we considered the van der Pol oscillator, modeled by the equation, $x'' + \mu(x^2 - 1)x' + x = 0$. Recall that there is a single critical point for this system, $x = 0$, near which we may model the behavior of the oscillator with the linear equation $x'' - \mu x' + x = 0$.

a. [5 points] Suppose that $\mu = -1$. Find the amplitude of the solution to the linear problem with initial condition $x(0) = 2$, $x'(0) = 3$. What is the time after which this amplitude never exceeds some value a_0?

b. [5 points] Suppose we force the linear system with an oscillatory input, so that we are considering $x'' - \mu x' + x = \cos(\omega t)$ (and $\omega \neq 0$). For what values of μ will the system have an oscillatory steady-state solution with frequency ω?

c. [5 points] Suppose that, for some choice of μ, the system $x'' - \mu x' + x = \cos(\omega t)$ has an oscillatory steady-state solution, and that the gain function $G(\omega)$ for this solution is shown to the right, below. If the steady-state solution to the problem is $y_{ss} = R\cos(t - \pi/2)$, what are the R in the solution, and ω in the forcing term? Why?



5. [14 points] In each of the following we consider a linear, second order, constant coefficient operator L, so that $L[y] = 0$ is a homogenous differential equation. (Note, however, that the operator L may be different in each of the parts below.) Let $y(0) = y_0$ and $y'(0) = v_0$, where y_0 and v_0 are real numbers.

a. [7 points] If the general solution to the equation $L[y] = 0$ is $y = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$, what is the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$?

b. [7 points] Now suppose that we are solving $L[y] = k$, for some constant k, and that y_0 and v_0 are both zero (so that $y(0) = y'(0) = 0$). If $Y(s) = \mathcal{L}\{y(t)\}$ is $Y = \frac{k}{s(s+3)(s+4)}$, what is the differential equation we are solving, and the general solution to the complementary homogeneous problem? Explain how you know your answer is correct.

6. [15 points] Each of the following concerns a linear, second order, constant coefficient differential equation $y'' + py' + qy = 0$.

a. [7 points] If the general solution to the problem is $y = c_1e^{2t}+c_2e^{4t}$, sketch a phase portrait for the system.

b. [8 points] Now suppose that for some real-valued α, we have $p = 2\alpha$ and $q = 1$, so that we are considering $y'' + 2\alpha y' + y = 0$. For what values of α, if any

(i) do all solutions to the differential equation decay to zero?

(ii) are there solutions that do not decay to zero?

(iii) will the general solution be a decaying sinusoidal function?

7. [12 points] For each of the following give an example, as indicated, and provide a short (one or two sentence) explanation of why your answer is correct.

a. [4 points] Give an example of an initial value problem with a linear, second-order, homogeneous differential equation for which there is no guarantee of a unique solution.

b. [4 points] Give an example of a linear, second-order, constant-coefficient, nonhomogeneous differential for which we cannot use the method of undetermined coefficients. What form will the general solution to your equation take?

c. [4 points] Give an example of a linear, second-order, nonhomogeneous differential equation for which the Laplace transform of the dependent variable y could be $\mathcal{L}\{y(t)\} = Y(s) = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$.

This page provided for additional work.

Formulas, Possibly Useful

- Some Taylor series, taken about $x = 0$:
  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
  $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  About $x = 1$: $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$.
- Some integration formulas: $\int u\,v'\,dt = u\,v - \int u'\,v\,dt$; thus $\int t\,e^t\,dt = t\,e^t - e^t + C$, $\int t\,\cos(t)\,dt = t\,\sin(t) + \cos(t) + C$, and $\int t\,\sin(t)\,dt = -t\,\cos(t) + \sin(t) + C$.
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.
- A coarse summary of partial fractions:

$$\frac{1}{(s+r_1)(s+r_2)^2((s+h)^2+k^2)} = \frac{A}{s+r_1} + \frac{B}{s+r_2} + \frac{C}{(s+r_2)^2} + \frac{D(s+h)+E}{(s+h)^2+k^2}.$$

Some Laplace Transforms

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$, $s > 0$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$
3.	t^n	$\frac{n!}{s^{n+1}}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
A.	$f'(t)$	$s\,F(s) - f(0)$
A.1	$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$
A.2	$f^{(n)}(t)$	$s^n F(s) - \cdots - f^{(n-1)}(0)$
B.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
C.	$e^{ct} f(t)$	$F(s-c)$